WHAT IS A FUNCTIONAL?

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1. Introduction. The use of the word "functional" (French—"fonctionnelle") to denote a function whose value depends on the form of an ordinary function of a real variable began with the work of Volterra and Fréchet about a half century ago. An example of such a functional is the length

$$L(y) = \int_a^b \sqrt{1 + y'^2}\, dx$$

of the graph of $y = y(x)$. Here the value of $L(y)$ depends on all the values taken by $y(x)$ on the interval (a, b). The same remark is true in general of the integrals

$$F(y) = \int_a^b \phi(x, y, y')dx$$

of the calculus of variations. The calculus of variations is in fact concerned with the theory of maxima and minima for a somewhat vaguely characterized class of functionals. In the general theory of functionals it is convenient to admit to consideration such special functionals as

$$F_1(y) = y(c),$$
$$F_2(y) = y'(c),$$

where c is a fixed point of the interval (a, b), as well as other more general types which are not considered in the calculus of variations.

The values taken by a functional may also be functions, as, for example,

$$z(x) = \int_a^b K(x, t, y(t))dt. \tag{1}$$

We may write this equation in the form $z = T(y)$, where we use $T(y|x)$ as a symbol for the right hand side of (1). Such functionals are sometimes called "functional transformations." Another example of a functional transformation is given by the familiar operation of taking the derivative of a function.

2. Functional equations. Many types of functional equations which may be used to determine functions are well known, and in fact their study makes up a large part of analysis. For example, we may consider an ordinary differential equation

$$y' = \phi(x, y)$$

with fixed initial condition $y(\xi) = \eta$. Then under suitable conditions on the function ϕ, the solution $y(x)$ is uniquely determined. This solution clearly depends on the form of ϕ, so we may write $y = F(\phi)$, and inquire into the properties of

the functional transformation $F(\phi)$. A theory of "differential corrections" based on such an inquiry was developed by Bliss [3, 4]. This theory is useful, for example, in computing the perturbations of the trajectories of projectiles, due to wind or other disturbances.

Another type of functional equation is the linear integral equation

$$y(x) = \int_a^b K(x, t)y(t)dt + z(x). \tag{2}$$

Here the solution $y(x)$ depends on both $z(x)$ and $K(x, t)$, so we are led to consider the functional transformation $y = F(z, K)$ giving the solution of (2) for each z and K for which it exists. In this case the transformation F is linear in the variable z.

3. Development of abstract function theory. The common features of various existence theorems in analysis suggest that a general and abstract theory, embracing many of them under one head, is possible. It is now generally recognized that an abstract theory has many advantages. For example, the existence and embedding theorems for ordinary differential equations become special cases of a general implicit function theorem. The title "functional analysis" is frequently used for an abstract theory of functions, as well as for theories of special functionals such as those described above. But the term "functional" has now become somewhat outmoded, and the term "function" is used to refer to a correspondence between variables of any type whatever.

The most widely used set of postulates for the basis of an abstract function theory characterize what is usually called a Banach space. Memoirs using essentially equivalent sets of postulates were published about the same time by Banach [1, 2], Hahn [7], and Wiener [14]. Another equivalent set may be stated as follows.

I. $\mathfrak{Y}$ is a class of elements, denoted by $y, y_1, y_2, \cdots$, and called points, which is a linear space over the real numbers. That is, (a) Addition of elements of $\mathfrak{Y}$ is defined, and $\mathfrak{Y}$ with addition forms a commutative group; (b) Scalar multiplication (meaning multiplication of elements of $\mathfrak{Y}$ by real numbers) is defined, and is associative with multiplication of real numbers, and distributive with respect to addition of real numbers and to addition of elements of $\mathfrak{Y}$.

II. Corresponding to each point y in $\mathfrak{Y}$ there is determined a real number $\|y\|$, called the norm of y, such that: (a) $\|y_1+y_2\| \leqq \|y_1\| + \|y_2\|$; (b) $\|ay\| = |a|\|y\|$; (c) $\|y\| = 0$ if and only if y is the identity element Θ of the group $(\mathfrak{Y}, +)$.

III. The space $\mathfrak{Y}$ is complete, in the sense that to every sequence (y_n) such that $\lim_{m=\infty, n=\infty} \|y_m - y_n\| = 0$, there corresponds a point y in $\mathfrak{Y}$ such that $\lim_{n=\infty} \|y_n - y\| = 0$. In other words, every Cauchy sequence in the space has a limit in the space.

The following examples of Banach spaces are very frequently useful.

1. The space $\mathfrak{E}_n$, which is the n-dimensional number space consisting of all n-uples $(y^1, \cdots, y^n)$ of real numbers. Here we may take $\|y\| = \max |y^i|$, or $\|y\| = (\sum |y^i|^2)^{1/2}$.

2. Real Hilbert space $\mathfrak{H}$, consisting of all infinite sequences (y^i) of real numbers such that $\|y\| = (\sum |y^i|^2)^{1/2}$ is finite.

3. The space $\mathfrak{L}_p$, corresponding to a fixed measurable set E and a number $p \geqq 1$, and consisting of all functions $y(t)$ which are measurable on E and have $|y(t)|^p$ Lebesgue-integrable over E. Here we set $\|y\| = [\int_E |y(t)|^p dt]^{1/p}$, and it becomes necessary to identify functions which are equal almost everywhere.

4. The space $\mathfrak{C}$, composed of all functions $y(t)$ which are continuous and bounded on a fixed set E. Here we set $\|y\| = \text{l.u.b.} |y(t)|$ on E. The set E is sometimes required to be bounded and closed, and may even be restricted to be an interval.

5. The space $\mathfrak{C}^{(n)}$, composed of all functions $y(t)$ which are continuous and bounded on a set E with suitable properties, and which have continuous and bounded derivatives up to and including those of the nth order. Here we may set $\|y\|$ equal to the least upper bound of $|y(t)|$ and of all its derivatives up to the nth order.

At the present time Hilbert space is also usually characterized postulationally. From this point of view the space $\mathfrak{L}_2$ becomes another realization of Hilbert space, on a par with $\mathfrak{H}$ as described under 2 above. It is clearly possible to admit functions with complex values in all the examples listed above. The scalar multipliers in the postulate system may then be allowed to be complex also.

Examples of functions defined on the spaces $\mathfrak{E}_n$ are familiar. The integrals of the calculus of variations mentioned above are functions defined on the space $\mathfrak{C}^{(1)}$, with real values. If we consider the transformation

$$z(x) = \int_a^b K(x, t)y(t)dt,$$

we see that it defines a function $z = F(y)$ on the space $\mathfrak{L}_2$, with values also in the space $\mathfrak{L}_2$ if $K(x, t)$ is in the space $\mathfrak{L}_2$ for the square $a \leqq x \leqq b$, $a \leqq t \leqq b$. If $K(x, t)$ is in the space $\mathfrak{C}$, then the values of $F(y)$ are also in the space $\mathfrak{C}$.

The generalizations of the fundamental ideas of function theory to abstract spaces, so far as these generalizations have been developed, naturally present a greater variety than is present in the classical function theory. For example, definitions of differentiability which coalesce for functions defined in the space $\mathfrak{E}_n$ turn out not to be equivalent in general. For brevity, we shall state only a definition of the class C' of functions defined on an open set Y_0 in a Banach space $\mathfrak{Y}$. Such a function $F(y)$ is said to be of class C' on Y_0 in case there exists a function $dF(y; \delta y)$ defined for each y in Y_0 and each δy in $\mathfrak{Y}$, with the properties: (a) $dF(y; \delta y)$ is linear and continuous in δy; (b) $dF(y; \delta y)$ is continuous in y uniformly for $\|\delta y\| = 1$; (c) for each y in Y_0.

$$\lim_{\|h\|=0} \frac{\|F(y + h) - F(y) - dF(y; h)\|}{\|h\|} = 0.$$

The function $dF(y; \delta y)$ is called the *differential* of F. When the space $\mathfrak{Y}$ is the Cartesian product of two Banach spaces $\mathfrak{W}$ and $\mathfrak{Z}$, that is, the variable y is a

symbol for the pair of variables (w, z), then the differential $dF(y; \delta y)$ is the sum of the partial differentials $d_w F(w, z; \delta w)$ and $d_z F(w, z; \delta z)$.

The ordinary theorem on the existence of functions defined implicitly has an immediate generalization in Banach spaces. See [8]. Let $F(y) = F(w, z)$ be of class C' on the open set Y_0, with values in the space $\mathfrak{W}$, and suppose that $F(w_0, z_0) = \Theta =$ the zero of the space $\mathfrak{W}$. Suppose also that the linear transformation $d_w F(w_0, z_0; \delta w)$ maps the space $\mathfrak{W}$ into itself in one-to-one fashion, that is, $w' = d_w F(w_0, z_0; \delta w)$ has an inverse transformation $\delta w = K_0(w')$. (This condition takes the place of the usual hypothesis on the non-vanishing of the Jacobian.) Then there is a unique function $G(z)$ defined and of class C' near z_0 and taking values near w_0, such that $F(G(z), z) = \Theta$. This theorem may be proved by the classical method of successive approximations applied to the equation

$$w = w - K_0(F(w, z)),$$

which is equivalent to $F(w, z) = \Theta$. It is interesting because such results as existence theorems for differential equations (including boundary value problems) and integral equations may be regarded as special cases of it. Moreover, various apparent extensions of the theorem itself become special cases of it when appropriate new variables are introduced.

As an application of the implicit function theorem, we may consider an integral equation of the form

$$y(x) = \int_a^b K(x, t, y(t))dt + z(x). \tag{3}$$

If we require the functions y and z to be in the class $\mathfrak{C}$, and suppose that $K(x, t, y)$ and its partial derivative $K_y(x, t, y)$ are continuous, and if equation (3) has an initial solution $(y_0(x), z_0(x))$ at which the linear integral equation

$$\eta(x) = \int_a^b K_y(x, t, y_0(t))\eta(t)dt + \zeta(x)$$

has a unique solution for $\eta(x)$, then the nonlinear equation (3) has a unique solution $y(x)$ near $y_0(x)$ for each $z(x)$ near $z_0(x)$. Furthermore, the solution $y = G(z)$ is of class C' as a function of z.

The general implicit function theorem may also be applied in certain cases to show the existence of solutions of differential equations satisfying boundary conditions. Let the equation be of the form

$$y'' = f(t, y, y'), \tag{4}$$

where f is of class C' for $a \leqq t \leqq b$, and for all values of y and y', and suppose the partial derivatives satisfy the inequalities

$$0 \leqq f_y(t, y, y') \leqq M,$$
$$|f_{y'}(t, y, y')| \leqq M,$$

for all (t, y, y'). Then the differential equation (4) has a solution taking arbitrary preassigned values A and B at the ends of the interval (a, b). Other applications include embedding theorems for use in the calculus of variations, in cases where a direct proof would be extremely involved, and theorems on elliptic partial differential equations.

4. Conclusion. The functional calculus has developed in many other directions, and no adequate notion of these developments and their ramifications can be conveyed here. As an additional sample, a theorem on the existence of a minimum for a function may be mentioned. As is well known, a function $f(y)$ defined and lower semi-continuous on a bounded closed set S in the space $\mathfrak{E}_n$ has a minimum, that is, an attained lower bound, on S. If we suppose that S is a set in an abstract Banach space $\mathfrak{B}$, the condition that S is bounded and closed is no longer sufficient. Instead, we may now require that S is compact on itself, that is, that every infinite subset of S has a point of accumulation in S. But this makes a theorem which is too restricted for applications in the calculus of variations. If we call a sequence (y_n) in S a *minimizing sequence* for f in S in case

$$\lim_{n=\infty} f(y_n) = \text{lower bound of } f \text{ on } S,$$

then we may state the following extension of the theorem on existence of a minimum:

THEOREM. *If $f(y)$ has a minimizing sequence (y_n) in S which is compact on S, and if $f(y)$ is lower semi-continuous relative to compact subsets of S, then f has a minimum on S.*

This theorem is valid in spaces more general than Banach spaces, and is only one of a number of theorems which it is interesting and useful to study under the least restrictive hypotheses.

Several books devoted to various aspects of functional analysis are included in the bibliography below. The number of memoirs on the subject is enormous, and the few works listed can give only an introduction to the field. Some of the numerous books and memoirs of Volterra include applications to hysteresis, phenomena connected with competing or coöperating populations, and phenomena of heredity.

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A CALL FOR REFORM IN HIGH SCHOOL MATHEMATICS*

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Down through the ages mathematics has been a highly respected subject, but in our present scientific, mechanistic, industrial civilization, mathematics is vastly more important than it ever was in any former age. Furthermore, the need for mathematics and trained mathematicians is constantly increasing, and this seems to be a permanent trend.

At a time, however, when mathematics is becoming constantly more necessary, and the talented student with deficient mathematical training is finding more and more doors leading to interesting professions closed to him, mathematics is being slowly but surely crowded out of our secondary curriculum by far less important subjects and activities. In a large portion of the country a pupil may graduate from high school with one year or less of mathematics.

High school mathematics is in a very unhealthy condition at the present time. When a doctor seeks to cure a patient, it is quite important for him to determine the cause of the malady.

Fifty years ago, mathematics was king of the required subjects; mathematics was supposed to develop, strengthen, and discipline the mind. However, early in the present century, educators and administrators began to question seriously the value of the mathematics then being taught. Teachers with little knowledge of mathematics or rigor were constantly insisting on mathematical rigor. Mortality in mathematics was very high; and as the great influx of pupils of lesser ability began, this became a serious problem. Finally the entire doctrine of formal discipline was swept away, and the mathematics based on this

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doctrine began to be supplemented by less difficult and more interesting subjects. This trend was augmented by the wave of soft pedagogy that began to sweep the country like a plague.

Unfortunately, leaders in mathematics did not first put their house in order before essaying its defense. As a result we have had a forty-year war between the educator and the mathematician, with mathematics constantly on the losing end. Like most wars, this has been unnecessary. More than fifty years ago the great leaders, Klein, Tannery, Perry, Borel, Eliot, Judd, Young, and Moore, pointed the way to reform that would have given us mathematics so easy to defend that no defense would have been necessary.

We can easily double the value of high school mathematics, make it far more interesting to our pupils and much easier to teach by heeding the advice given by E. H. Moore in 1903 and by dozens of other great leaders since his time. Among these leaders are Myers, Breslich, Smith, Reeve, Schorling, Betz, Clark, Hedrick, and Swenson. But as Hedrick said shortly before his death, "Everybody renders lip service to the reform movement, but nobody does anything about it."

To be brief at the expense of a certain amount of oversimplification, the reforms needed are as follows:

1. We must stop trying to teach mathematics by the stupid "water-tight compartment" method long since abandoned or greatly modified by practically all other countries. The now classic report of the Association's 1923 Committee on Mathematical Requirements says: "The results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of extreme 'water-tight compartment' methods of presentation."

W. D. Reeve says, "Our traditional 'water-tight compartment' method of teaching algebra, then geometry, then intermediate algebra, leads to a great deal of unnecessary repetition of subject matter that results in the loss of time and energy" [1]. B. Bradford repeats the charge with the statement, "Of the many vices characteristic of much mathematical education not the least is the separation of the science into a number of almost water-tight compartments" [2]. W. Betz says, "The prevailing curriculum ignores not only the organic relationships of the various branches of mathematics but also their fruitful application. Instead we still insist on studying each mathematical subject in a compartment by itself and in a manner which is neither psychological nor practical. Moreover, we crowd each of these compartmentalized courses into a single semester or a single year" [3]. And, from D. E. Smith comes the accusation, "Of all the advanced countries of Europe and America, and we may also include Japan, we have been unquestionably the most backward in our introductory work in mathematics" [4].

2. We must remove the large amount of relatively useless material that is found in our best modern texts, and introduce more interesting and more practical mathematics to take the place of the dead wood eliminated.

More than twenty-five years ago a committee of the Association reported:

"The situation that needs to be met may best be illustrated by the case of algebra. Our elementary algebra is, in theory and symbolism, substantially what it was in the seventeenth century. The present standards of drill work, largely on non-essentials, were set up about fifty years ago. A considerable number of teachers, both in the secondary schools and the colleges, believe that the amount of time spent by pupils on abstract work in difficult problems in division, factoring, fractions, simultaneous equations, radicals, *et cetera*, is excessive; that such work leads to nothing important in the science, and adds little to facility in the manipulation of algebraic forms." This was quoted in the recent report of the National Council of Teachers of Mathematics Commission on Post War Plans and is one of the highlights of the report. W. D. Reeve says, "We shall have to get rid of the deadwood, the debris that clutters up the courses in mathematics" [1]. Raleigh Schorling also asserts, "No one should assume that all is well with the traditional courses. In most schools they are woefully out-of-date as regards both subject matter and method" [5].

3. We must have ability grouping in mathematics. There must be at least two and at best three types or levels of mathematics in high schools.

For "track" one we should have four years of the best type of "integrated" mathematics or some form of superior "general mathematics" for our very best students. Unfortunately, the term "general mathematics" has come to mean inferior or non-college mathematics. Other terms used for the original meaning are "fused," "correlated," "integrated," and "related." This is the kind of mathematics taught in all countries outside the United States.

For "track" two we should have three years of highly practical mathematics including at least a year of social and economic mathematics.

For "track" three we should have, where possible, a third track of two years of work similar to that of track two but less difficult.

We have always had ability grouping in athletics. If the results in the teaching of mathematics were as immediately determined as in athletics, we should have ability grouping at once. Ability grouping is the essence of true democracy, that is, each pupil entitled to be developed to his limit without being held back by less capable pupils.

Unfortunately, ability grouping has been tried in many places without much success. We must have texts, courses of study, and methods for each level, or much of the possible efficiency of ability grouping will be lost. We have never had such texts.

In the January *Mathematics Teacher*, Professor Reeve says, "How can the content be so reorganized and taught as to be worth while in the lives of the boys and girls who make up the secondary school? It is time for action." And in a recent letter from which I have received permission to quote, Professor Wren says, "I have a matter I want to present to you for your consideration. I have been thinking for some time that there is a big problem which the National Council and the Association ought to attack together. It is a reconsidera-

tion of the problem of the curriculum and instruction in mathematics, including teacher training, for grades 1 through 14. We need a strong committee, representative of the best thought in the organizations. It should be given money with which to operate, with the thought in mind that it should think through its problem and frame its procedure in such a way that an appeal might be made for support from some foundation such as the General Education Board or the Carnegie Foundation. This is really a major problem of great import to the schools. The personnel of the committee should be composed of individuals who have a sincere interest in mathematics but who are not so steeped in traditionalism that they cannot approach the problem with an open mind. We need to weed out some of the errors and stupidity of traditionalism and shape a program which is directed more nearly toward the interests and welfare of the pupils concerned."

No one can see the problem and its possible solution more clearly than Professor Wren. If the problem can be solved, it will be worth many millions of dollars yearly to the country and will be one of the most important projects in the history of American education. To secure the needed and long overdue reforms in mathematics we must have a committee of experts with adequate financing, so that they may devote their full time to this most important project. This may well take a year or more. The final report should be printed and given wide distribution. Part of this report could be a set of specimen or model textbooks from which textbooks authors could use any material they desired.

Nothing is so important to a nation, school, association, or factory as a plentiful supply of the best possible raw material. We can never expect to get adequate raw material—pupils with the mathematical training the colleges desire—until we have solved the problem Professor Wren poses.

At the present time several state curriculum committees are studying the reorganization of mathematics. There is very little hope that they will materially improve the situation in mathematics. A few tired teachers with heavy teaching schedules cannot do the work that Professor Wren and the author believe to be beyond the financial ability of such strong associations as the Mathematical Association of America and the National Council of Teachers of Mathematics. It is like sending a child with a trowel to do the work of a powerful bulldozer. In fact, considerable harm may be done by such state committees, should they succeed in convincing the teachers in their states that they have solved the problem.

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ON THE PARTITION OF NUMBERS INTO SQUARES

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1. Introduction. Let us denote by $P_k(n)$ the number of partitions of n into k integral squares $\geqq 0$. The term partition implies that we do not reckon as distinct two decompositions of n into k squares in which the squares are merely permuted. For example, since

$$\begin{aligned} 98 &= 9^2 + 4^2 + 1^2 \\ &= 9^2 + 3^2 + 2^2 + 2^2 \\ &= 8^2 + 5^2 + 3^2 \\ &= 8^2 + 4^2 + 3^2 + 3^2 \\ &= 7^2 + 7^2 \\ &= 7^2 + 6^2 + 3^2 + 2^2 \\ &= 6^2 + 6^2 + 5^2 + 1^2 \end{aligned}$$

we have $P_1(98) = 0$, $P_2(98) = 1$, $P_3(98) = 3$ and $P_4(98) = 7$.

The function $P_k(n)$ is practically unknown in spite of the extensive literature on sum of squares, and it appears to be a function of great complexity. The simplest question one can think of is that of solving the equation $P_k(n) = 0$.

This problem is completely solved. Every number, according to Fermat's theorem, is the sum of four squares, so that for $k \geqq 4$, $P_k(n) > 0$. For $k = 3$, every number, except all those of the form $4^a(8m+7)$, is a sum of three squares. For $k = 2$, $P_2(n) = 0$ if and only if the highest power of a prime $p = 4x+3$ dividing n is odd. For $k = 1$, $P_1(n) = 0$ only for non-squares.

In this note we attempt the modest question of solving the equation

$$P_k(n) = 1,$$

that is, of finding all integers n which are sums of k squares in essentially one way. We shall see that this problem is almost solvable. It is comparatively easy to write down generating functions for $P_k(n)$. One such is

$$k! \sum_{n=0}^{\infty} P_k(n) x^n = \frac{\partial^k}{\partial z^k} \left\{ (1 - z)(1 - xz)(1 - x^4 z)(1 - x^9 z) \cdots \right\}^{-1} \Big]_{z=0}.$$

However this is not a very useful tool for the direct attack on $P_k(n)$ since the development of the right member in powers of x is best accomplished by actually partitioning each number n into squares.

2. Partition into 4 squares. We begin with the case of $k = 4$. The question of the number of ways in which every number is the sum of four squares was answered by Jacobi [1] in 1828 by the following remarkable theorem:

JACOBI'S THEOREM. *The number $R_4(n)$ of representations of n as the sum of four squares is given by*

$$R_4(n) = 8[2 + (-1)^n]\sigma_0(n),$$

where $\sigma_0(n)$ denotes the sum of the odd divisors of n.

This beautiful result would seem to leave nothing to be desired. Thus for $n=98$ we have

$$R_4(98) = 8\cdot 3\cdot \sigma_0(98) = 8\cdot 3\cdot (1 + 7 + 49) = 1368.$$

However, as we have seen, $P_4(98)=7$. The explanation of this apparent discrepancy lies, of course, in the distinction between "partition" and "representation." For the problem of 4 squares the latter term implies that all permutations of the summands and both signs of the square roots of the squares >0 are reckoned as distinct solutions of the problem of making up the desired sum. To put the same thing in geometrical language, $R_4(n)$ is the number of lattice points in four-dimensional space on the surface of the hypersphere of radius $n^{\frac{1}{2}}$ about the origin. To the uninitiated student there appears to be only 7 "really" distinct solutions and to him this note is dedicated. To him it is a fair question to ask how one may obtain the reasonable number 7 from the obviously inflated value 1368. This problem is unfortunately one of extreme difficulty. At least no solution has ever been given. The difficulty lies in the unequal number of representations derivable from a single partition. It is not difficult to see that there are eleven different types of partitions as far as we are concerned. These may be listed, each with the number of derivable representations, as follows, where a, b, c, d represent different non-zero integers.

(1)

I	$a^2 + b^2 + c^2 + d^2$	384
II	$0^2 + a^2 + b^2 + c^2$	192
III	$a^2 + a^2 + b^2 + c^2$	192
IV	$a^2 + a^2 + b^2 + b^2$	96
V	$0^2 + a^2 + a^2 + b^2$	96
VI	$a^2 + a^2 + a^2 + b^2$	64
VII	$0^2 + 0^2 + a^2 + b^2$	48
VIII	$0^2 + a^2 + a^2 + a^2$	32
IX	$0^2 + 0^2 + a^2 + a^2$	24
X	$a^2 + a^2 + a^2 + a^2$	16
XI	$0^2 + 0^2 + 0^2 + a^2$	8

Thus the above 7 partitions of 98 into squares are of the types II, III, II, III, IX, I, III, respectively, and we find that

$$192 + 192 + 192 + 192 + 24 + 384 + 192 = 1368$$

as predicted by Jacobi's Theorem.

A fundamental relation for the function $P_4(n)$ is

$$P_4(2n) = P_4(8n). \tag{2}$$

This follows at once from noting that any partition of $8n$ into 4 squares $\geqq 0$ must involve even squares only. Hence there is a one to one correspondence between the partitions of

$$2n = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

and

$$8n = (2x_1)^2 + (2x_2)^2 + (2x_3)^2 + (2x_4)^2.$$

By a similar argument it is seen that

$$P_4(4n) = P_4(n) + P_4'(4n)$$

where $P_4'(n)$ is the number of partitions of n into 4 odd squares. This relation contains (2) as a special case when n is even. Formula (2) shows that we may confine our problem of evaluating $P_4(n)$ to the case in which n is not divisible by 8.

Inequalities for $P_4(n)$ follow at once from Jacobi's theorem and from (1) which shows that

$$R_4(n)/384 \leqq P_4(n) \leqq R_4(n)/8.$$

Thus if ω is an odd number,

$$P_4(\omega) \geqq \sigma(\omega)/48 \tag{3}$$

$$P_4(2\omega) \geqq \sigma(\omega)/16 \tag{4}$$

$$P_4(4\omega) \geqq \sigma(\omega)/16 \tag{5}$$

where $\sigma(\omega)$ is the sum of all the divisors of ω.

We can prove the following theorems.

THEOREM 1. *The numbers* 1, 3, 5, 7, 11, 15, 23 and $4^\alpha r$, $r = 2$, 6, and 14, $\alpha \geqq 0$, *and no others may be partitioned into* 4 *squares* $\geqq 0$ *in only one way.*

THEOREM 2. *The numbers* 9, 13, 17, 21, 29, 31, 35, 39, 47, 71 *and* $5 \cdot 2^\alpha$, $11 \cdot 2^\alpha$, $M \cdot 4^\alpha (M = 1, 3, 30, 46)$, $\alpha > 0$, *and no others may be partitioned into* 4 *squares* $\geqq 0$ *in precisely two ways.*

A recent table of Gino Loria [2] gives partitions of n into four squares $\geqq 0$ for all $n \leqq 100$. This makes the proofs of Theorems 1 and 2 very easy. In fact we may invert this table after correcting its errata [3] and list for each $r \geqq 1$ the values of $n \leqq 100$ for which $P_4(n) = r$ as follows:

r	
1	1, 2, 3, 5, 6, 7, 8, 11, 14, 15, 23, 24, 32, 56, 96
2	4, 9, 10, 12, 13, 16, 17, 19, 20, 21, 22, 29, 30, 31, 35, 39, 40, 44, 46, 47, 48, 64, 71, 80, 88
3	18, 25, 26, 27, 28, 33, 37, 38, 41, 43, 51, 53, 55, 59, 60, 62, 72, 79, 92, 95
4	34, 36, 42, 45, 49, 57, 61, 63, 65, 67, 68, 69, 77, 78, 83, 87, 94
5	50, 52, 54, 58, 70, 73, 74, 75, 76, 84, 85, 86, 89, 91, 93
6	66, 81, 97, 99
7	82, 98, 100
8	—
9	90

To prove Theorem 1 we note that inequalities (3), (4), and (5) imply

$$P_4(n) \geqq \frac{\sigma(n)}{48} > \frac{n}{48} \qquad (n \neq 4m)$$

$$P_4(n) \geqq \frac{\sigma\left(\frac{n}{4}\right)}{16} > \frac{n}{64} \qquad (n = 4\omega).$$

In order that $P_4(n) = 1$, it is therefore necessary that $n < 64$ if n is not a multiple of 8. Hence, excepting multiples of 8, inspection of the above table shows that $n = 1, 2, 3, 5, 6, 7, 11, 14, 15$, and 23. If n is a multiple of 8, write $n = 4^\alpha \cdot 2m$, where $2m$ is not a multiple of 8. Then by α applications of (2),

$$P_4(n) = P_4(2m).$$

In order that $P_4(n) = 1$ it is therefore necessary and sufficient that $2m = 2, 6, 14$. Thus the proof of Theorem 1 is complete.

Theorem 2 may be proved in the same general way.

The case of $k > 4$. We now consider the case of partitions into 5 squares. Let n be an integer for which $P_5(n) = 1$. Suppose first that $n \geqq 16$. Then each of the integers

$$n - 0^2, \quad n - 1^2, \quad n - 2^2, \quad n - 3^2, \text{ and } n - 4^2$$

is non-negative and hence is the sum of 4 squares. Each of these partitions into 4 squares implies a partition of n itself into 5 squares of which one of the squares implies a partition of n itself into 5 squares of which one of the squares is 0^2, 1^2, 2^2, 3^2, and 4^2, respectively. Since n is supposed to possess but one such partition it follows that

$$n = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

But also $30 = 3^2 + 3^2 + 2^2 + 2^2 + 2^2$ so that $P_5(30) > 1$. Hence we must look for n among the first 15 integers. By Theorem 1 we narrow the search to $n = 1, 2, 3, 5, 6, 7, 8, 11, 14$ and 15. But 5, 8, 11 and 14 are each sums of 5 non-zero squares. Hence $P_5(n) = 1$ if and only if $n = 1, 2, 3, 6, 7$ and 15.

Now 6 and 15 are sums of 6 non-zero squares so that $P_6(n) = 1$ if and only if $n = 1, 2, 3$, and 7. Finally for $k > 6$, $P_k(n) = 1$ if and only if $n = 1, 2$ and 3.

The case of $k<4$. It remains to consider the cases $k=1$, 2 and 3. The first of these is trivial since $P_1(n)=1$ or 0 according as n is a square or not. By a theorem announced by Fermat [4] and proved by Euler it follows that $P_2(n)=1$ if and only if n is of one of the forms

$$n = 2^{\alpha}Q^2 \quad \text{or} \quad n = 2^{\alpha}Q^2p,$$

where in either case $\alpha \geqq 0$, Q is an odd integer having no prime factor of the form $4x+1$, and where p is a prime of the form $4x+1$.

Finally we take up the case $k=3$. It is easy to see that there is a one-to-one correspondence between the solutions of $x^2+y^2+z^2=n$ and of $(2x)^2+(2y)^2+(2z)^2=4n$. Hence we may lay aside those n's which are multiples of 4. Also $n=8x-1$ is not sum of three squares. For all other values of n the number $R_3(n)$ of representations of n as a sum of three squares is expressible in various ways [5] in terms of the number of classes of binary quadratic forms of determinant $-n$. Here we find that in order to get this result one uses only non-negative values of x, y, z in $x^2+y^2+z^2=n$ (as opposed to the case of four squares). All permutations of x, y, z are counted. There are thus only the following three types of partitions with corresponding numbers of representations

I	$a^2+b^2+c^2$	6
II	$a^2+a^2+b^2$	3
III	$a^2+a^2+a^2$	1

where a, b, c, are distinct integers $\geqq 0$. What prevents us from repeating the type of argument used in solving $P_4(n)=1$ is the absence of a good inequality for the class number function similar to $\sigma(n)>n$. Recent researches [6] have shown that the class number function assumes a given value only finitely often, as conjectured by Gauss. Hence the equation $P_3(n)=1$ has only a finite number of solutions which are not multiples of 4.

A recent table of Gupta [7] gives the number of representations $R_3(n)$ for all $n \leqq 10000$. In order that $P_3(n)=1$, it is necessary but not sufficient that $R_3(n)=1$, 3, or 6. By examining all cases where $R_3(n)$ has these values we are led to the following theorem.

THEOREM 3. *The equation* $P_3(n)=1$ *has the following solutions:* $n=4^{\alpha}m$, *where* $\alpha \geqq 0$, *and* $m=1$, 2, 3, 5, 6, 10, 11, 13, 14, 19, 21, 22, 30, 35, 37, 42, 43, 46, 58, 67, 70, 78, 91, 93, 115, 133, 142, 162, 182, 190, 235, 253, 403, 427. *Any further values of* m *lie beyond* 10000.

It is extremely likely that the above list of solutions of $P_3(n)=1$ is complete. An actual proof of this fact however must await more precise information about the class number of binary quadratic forms.

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THE ACTUARIAL PROFESSION

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1. Character of actuarial work. This article has been prepared to furnish college students with a brief description of the character of actuarial work and an outline of the opportunities, inasmuch as there is a great shortage of young men with the mathematical ability, knowledge, and qualities of business leadership required in the actuarial profession.

It is the actuary who is responsible for calculating the premiums a life insurance company must charge and it is he who prepares the tables of death rates upon which such calculations are based. In actual practice his duties cover a much wider field than such technical responsibilities. They include the decision as to what benefits shall be contained in life insurance policies and how much money must be set aside from year to year to guarantee the payment of such benefits many years in the future. The actuary must analyze the sources of earnings under policy contracts so that he may determine proper rates of dividends. He investigates the effect on mortality of various physical impairments, hazardous occupations, and other unusual risks, and in collaboration with the medical officer determines the basis for accepting or rejecting applicants for insurance. Because of his broad fundamental training, the actuary of a life insurance company usually has an important part in developing the general executive policies of the company. Although he cannot operate without a thorough knowledge of the mathematical basis of life insurance, he is more of a business man than a mathematician. Not the least of his duties is to explain complicated problems to other business men and to policyholders in language clear enough to be readily understood.

The great changes in our social and economic structure that have taken place in recent years have brought about problems requiring the special attention of the actuary. The severe reduction in the rate of interest realized on new investments has required a general revision of premium rates as well as comprehen-

sive long-term planning to provide adequately for the security of policies issued many years ago when interest rates were high. The development of social insurance of various types and, in particular, the programs of compulsory health and accident insurance that have been enacted by several states and have been proposed in a number of others, have posed many new problems in mathematical and statistical theory and in practical administration. It is largely the responsibility of the actuary to solve these problems.

A prominent contemporary actuary has described the qualities required for success in actuarial work as "competent statistical and mathematical capacity, adequate economic and financial knowledge, and wide social information," and the successful actuary has been further characterized as a person "with a determined, lively, and ingenious mind and a broad outlook." Although the actuarial profession is not restricted to the single field of life insurance, the majority of actuaries are to be found in life insurance companies. Some actuaries are employed by the Federal and state governments, and some are established as independent consultants.

2. Employment and opportunities. Employment by life insurance companies, or other employers, of persons who wish to enter the actuarial profession depends upon the individual choice of the employer. The first requirement is that the candidate give evidence that he will probably be able to pass the examinations described in Section 4. While a very good college record in mathematics is usually considered satisfactory evidence to this effect, the candidate who has received credit for some of the preliminary actuarial examinations taken while in college will have an advantage. After the candidate has given satisfactory evidence of his mathematical competence, the employing company generally has him interviewed by several individuals who attempt to assess his abilities on the basis of the non-mathematical qualities required of an actuary. Even though there may be no question as to his technical mathematical ability, he is employed only if the result of these interviews is favorable.

The actuarial profession is not crowded. There are more than 500 life insurance companies in the United States and Canada. Most of these companies are growing in size and, as is the case with other activities in our civilization, their business is continually becoming more complex. As a result of the wide scope of actuarial training the actuary is particularly well fitted to secure a proper perspective of the interrelated problems of a life insurance company and his advice is much sought after on questions of general company policy.

All the life insurance companies need the services of actuaries. The large companies, particularly those that are engaged in writing industrial insurance and group insurance, need large staffs of actuaries. Moreover, there is a growing tendency for men initially trained as actuaries to move on to other spheres of activity in the insurance offices, particularly in the investment, administrative, underwriting, and accounting fields. A considerable number of actuaries fill high executive positions.

There are many needs for actuaries outside of the life insurance companies.

The increasing trend towards pension and retirement plans in business and industry provides many opportunities in the various firms of consulting actuaries. There are actuarial positions in many of the state insurance departments, which supervise and regulate the insurance business. The Federal governments of both the United States and Canada use actuaries in several departments, as for example in the Social Security Administration, the Veterans Administration, and the National Office of Vital Statistics. There are occasional calls from Latin America and other foreign countries for American-trained actuaries.

An actuarial career has the advantage that adequate salaries are paid even during the training period. Later on, the actuary's salary naturally depends upon the extent of the responsibilities he is capable of assuming. The average salary earned by qualified actuaries is large, because they are in considerable demand as highly trained specialists. There are only about 600 persons who are Fellows of The Actuarial Society of America or the American Institute of Actuaries.

3. Qualification as an actuary. Qualification is attained by passing a number of examinations required for membership in a recognized actuarial organization. In the life insurance field in the United States and Canada, these organizations are The Actuarial Society of America and the American Institute of Actuaries. Both of these societies are private organizations run entirely by their members and have been organized for the purpose of promoting the theory and practice of actuarial science. Through their publications and the regular meetings of the members, there is an interchange and development of professional knowledge, as well as the formation of congenial friendships.

Admission to membership is by a series of examinations given jointly by the two societies. These examinations are described in Section 4. On passing a series of five examinations the candidate becomes an Associate member, and on passing three more examinations a full member or Fellow. Ordinarily the later examinations are taken by the candidate while he is employed by a life insurance company or in some other actuarial capacity, since practical experience is very helpful, in fact almost essential, in preparing for these examinations. It is a great advantage to the applicant, however, to be able to take the earlier examinations while he is still in college, as they cover subjects which are ordinarily taught in colleges. Another advantage is that if the student starts his examinations while still in college he can measure his aptitudes and probability of passing and thus determine before he graduates whether to pursue an actuarial career or to enter some other field.

4. Preliminary examinations.* There are three examinations, given usually

* These three examinations are conducted by the Educational Testing Service (which, since January 1, 1948, has taken over the testing activities of the College Entrance Examination Board), P.O. Box 592, Princeton, New Jersey. An application form showing the exact dates of the examinations, together with a pamphlet giving an outline of the subject matter to be covered and illustrative questions, can be secured by applying to The Actuarial Society of America, 393 Seventh Avenue, New York 1, New York, or the American Institute of Actuaries, 135 South La Salle Street, Chicago 3, Illinois.

in May of each year, which it is well for the candidate to take while he is in college. The following is a description of the present syllabus upon which the examinations are based:

Language Aptitude Examination. This is a three-hour aptitude examination on reading comprehension and precise knowledge of the meaning of words and of word relationships. It is similar to the well known Scholastic Aptitude Test of the College Entrance Examination Board, except that it is pitched at approximately the college sophomore level. This is not the type of examination for which specific preparation can be made; it is an aptitude rather than an achievement examination.

Residents of Canada who take the examination at a Canadian center may take a French form of Part 1, in lieu of the English form, if they indicate their desire to do so on the application form when it is submitted.

This examination will normally be taken by the undergraduate at the end of his sophomore year in college, but it is not restricted to college sophomores.

General Mathematics Examination. This is a three-hour achievement examination based on the material usually covered in the first two years of mathematics in American and Canadian colleges and universities. Although the candidate is expected to have mastered the elements of trigonometry and coördinate geometry, the emphasis is on college algebra and differential and integral calculus. Students who have not had courses in these latter subjects recently should review them very thoroughly.

The following list of topics will serve to indicate the general scope of the examination and the point of view from which the material is to be considered rather than to provide a complete catalogue of topics:

(a) *Algebra.* Theory and practice of logarithms; variations; simultaneous linear equations and other simple simultaneous equations; progressions; binomial theorem; permutations and combinations; elementary theory of equations.

(b) *Trigonometry, Coördinate Geometry and Calculus.* Trigonometric functions and identities; equations of the straight line, circle, parabola, ellipse, and hyperbola; functions and limits; derivatives of the standard elementary functions; applications of derivatives; partial differentiation; integration; applications of the definite integral, including simple multiple integrals; expansion of functions into series. Approximately one half of the examination time will be devoted to algebra. This examination will normally be taken by the mathematically talented undergraduate at the end of his sophomore year.

Special Mathematics Examination. This is a three-hour achievement examination based on the material usually covered in undergraduate courses in finite differences, probability and statistics. It can be taken by college students who have had courses in these subjects or by persons who have done concentrated reading in these fields. The following list of topics will serve to indicate the general scope of the examination and the point of view from which the material

is to be considered rather than to provide a complete catalogue of topics:

(a) *Finite Differences.* Operations with symbols; interpolation for equal and unequal intervals; inverse interpolation; simple interpolation for functions of two variables; summation and approximate integration. (The student is expected to be familiar with the advancing, central, and divided difference notations.)

(b) *Probability and Statistics.* Addition and multiplication laws of probability, compound probability; permutations and combinations and their application to probability problems; binomial, Poisson, normal, and other probability distribution functions; statistical frequency functions; means, standard deviations, and moments of probability distributions and frequency functions; probability and frequency distributions in two variables; simple correlation and regression; sampling and sampling errors of means, proportions and percentages; simple tests of significance. Approximately one half of the examination time will be devoted to finite differences.

5. Subjects to be taken in college. The student should prepare for the different mathematical subjects on which the preliminary examinations are given; usually his major will be mathematics as a matter of course, so that he will secure a broad mathematical foundation. As in all occupations requiring judgment and a broad cultural background, a thorough grounding in English composition is essential. The student would benefit from a course in general business accounting, if courses are available in that subject, as it is an important aid in understanding life insurance accounts, both for the later actuarial examinations and for actuarial practice. If he has the opportunity, the student should also take at least one full-year course in economics. Aside from these particular subjects, it is most important for the candidate to study subjects which will give him a broad cultural foundation, as his most important responsibility will be the exercise of sound judgment.

6. Later examinations. In addition to the Preliminary Actuarial Examinations, the student must pass Parts 4 and 5 if he is to become an Associate of The Actuarial Society of America or the American Institute of Actuaries, and before taking these examinations he must obtain the approval of the Society or the Institute, as the case may be, to a formal application for Associateship. In order to become a Fellow he must also pass Parts 6, 7 and 8 of the actuarial examinations.

The list of subjects covered by Parts 4 to 8 of the actuarial examinations is shown below. These examinations are given jointly by the Society and the Institute, and candidates intending to prepare for them can obtain the recommended course of reading by writing the Society or the Institute. There are a few colleges in the United States and Canada that give courses in some of these advanced actuarial subjects.

Associateship. Parts 4 and 5, with the subjects and approximate date of the examinations, are:

Part 4, six hours, given about May 1:

(a) Compound interest and annuities-certain.
(b) The mathematics of life contingencies.
(c) Multiple decrement tables.
(d) Computation of tables involving (a), (b), or (c).

Part 5, six hours, given shortly before December 1:

(a) Construction of mortality and disability tables, including the elements of graduation.
(b) The sources and characteristics of the principal mortality and disability tables (including the methods used in the construction and graduation of such tables) and of the principal mortality and disability investigations.
(c) Selection of risks, including underwriting of disability and additional accidental death benefits, insurance of under-average lives and premiums for extra hazards.

Fellowship. Parts 6 to 8, with the subjects and approximate date of the examinations, are:

Part 6, six hours, given about May 1:

(a) Gross premiums for insurance, annuities, disability, and other benefits.
(b) Valuation of the liabilities of life insurance organizations.
(c) Nonforfeiture values and change or reinstatement of life insurance contracts.

Part 7, six hours, given shortly before December 1:

(a) Analysis and distribution of surplus.
(b) Life insurance accounting.
(c) Investment of life insurance funds and valuation of assets.

Part 8, six hours, given about May 1:

(a) Life insurance law.
(b) Optional subjects. (Each student, at the time of payment of the fee covering Part 8, will be required to choose two of the following five subjects. In the examination room he will receive the papers for those two subjects but not for the other three. A student may change his choice by written notice to the Secretary before March 15 preceding the examination.)
    (1) Advanced graduation and interpolation: advanced mathematical theories and formulas excluded from Part 5(a).
    (2) Employee retirement plans: actuarial phases of self-insured pension funds; group annuities; essential principles of United States and Canadian income tax laws and regulations as they apply to retirement plans and contributions and benefits thereunder.
    (3) Actuarial phases of agency problems.
    (4) Group insurance; group life and group health insurance, including

hospitalization and surgical and medical reimbursement benefits.
(5) Social insurance: fundamental principles; existing social insurance plans in the United States and Canada.

MATHEMATICAL NOTES

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CASSINI OVALS ASSOCIATED WITH A SECOND ORDER MATRIX

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In a recent paper [1], Brauer proved that, if

$$P_k = \sum_{j=1,\, j\neq k}^{n} |a_{kj}|, \qquad k = 1, 2, \cdots, n,$$

then each characteristic root, λ, of a matrix $A = (a_{ij})$ lies in the interior or on the boundary of at least one of the $n(n-1)/2$ Cassini ovals:

$$|z - a_{kk}|\,|z - a_{mm}| = P_k P_m.$$

In case $n=2$, there is only one oval and the characteristic roots are on its boundary. The purpose of this note is to exhibit a relationship between the Cassini oval thus defined for a second order matrix and the field of values of the matrix, used elsewhere as the basis of theorems on limits for the characteristic roots [2].

Consider the set M of matrices which are unitary transforms of a given second order matrix A (i.e., the matrices $UA\overline{U}'$, where $U\overline{U}' = I$). These matrices have the same characteristic roots as A and the same field of values, an ellipse whose foci are the characteristic roots [3]. Since for each matrix of M there is a unique Cassini oval there is generated a set, C, of Cassini ovals associated with the fixed ellipse. Each Cassini oval of C has as its foci the principal diagonal elements of the corresponding matrix of M. Now the principal diagonal elements of a matrix belong to its field of values, and, since

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i,$$

they have the same centroid as the characteristic roots. Therefore, the foci of any Cassini oval of C are a pair of points within or on the ellipse and sym-

metrically placed with respect to its center. Conversely, each two such points, p and q, are the foci of some curve of C, for, to obtain the corresponding matrix of M, it suffices to take as the first row of U a unit vector u such that $uA\bar{u}'=p$.

The idea of comparing the Cassini ovals of C with the field of values as limits for the characteristic roots led to the following result.

THEOREM. *The set of points covered by the interiors and boundaries of the Cassini ovals of C consists of the interior and circumference of the circle concentric with the ellipse and with radius* $[a^2+b^2]^{1/2}$, *where a is the semi-major axis, and b the semi-minor axis, of the ellipse.*

The ellipse may be assumed in standard position. Evidently the Cassini ovals which extend farthest from the center are those whose foci are on the circumference of the ellipse. Consider, then, the sub-family C' whose foci are at (u, v) and $(-u, -v)$ for $u^2/a^2+v^2/b^2=1$. Let r and s be the constants of the Cassini oval; thus, r is the distance from the center to a focus, and s^2 is the product of the distances from a point on the oval to the two foci. Then

$$r^2 = u^2 + v^2 = \frac{a^2b^2 + c^2u^2}{a^2},$$

where c is the distance from the center to a focus of the ellipse.

Since the oval passes through the foci of the ellipse, we have

$$s^2 = [(c-u)^2 + v^2]^{1/2}[(c+u)^2 + v^2]^{1/2} = \frac{a^4 - c^2u^2}{a^2}.$$

On adding the above results, we obtain

$$r^2 + s^2 = a^2 + b^2.$$

But $[r^2+s^2]^{1/2}$ is the distance from the origin to the end of the major axis of the Cassini oval, so that this distance, for members of C', is constantly $[a^2+b^2]^{1/2}$. Furthermore, all other points of the oval are closer to the origin. The fact that C covers the circle of the theorem is now evident, as each point in or on the ellipse is a focus for some oval of C, and hence certainly interior to it, and each point outside the ellipse but inside or on the circle lies on the line joining the focus of some member of C' to the end of its major axis.

The diagonals of the circumscribing rectangle of the ellipse with sides parallel to the axes of the ellipse are diameters of the circle of the theorem, and serve to classify the ovals of C' into those with two loops, the lemniscates, and those with one loop, according as the foci are in the smaller angle formed by the diagonals, on a diagonal, or in the larger angle.

Some special cases are of interest. There are some matrices in M whose diagonal elements are equal; the associated Cassini ovals are degenerate, being circles with the line joining the foci of the ellipse as diameter. If $b=0$, the ellipse degenerates to a straight line segment, and the circle of the theorem has this line

segment as diameter. If $c=0$, the ellipse is a circle, and the associated Cassini ovals are all lemniscates.

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A SYMBOLIC FORM OF AN INVERSION FORMULA FOR A LAPLACE TRANSFORM

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In a paper which appeared in this journal [1], W. C. G. Fraser proved the following theorem

THEOREM A. *If $\alpha(t)$ is a function of bounded variation in every finite interval $(1, R)$ with $\alpha(1)=0$, and if*

$$f(s) = \int_1^\infty t^{-s}\,d\alpha(t), \tag{1}$$

the integral converging for some value of s, then

$$\alpha(t) = \lim_{s\to+\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\, t^{ks} f(ks) \tag{2}$$

at points t where $\alpha(t)$ is continuous.

As Fraser pointed out, this is an elaboration of an earlier result of E. Phragmén [2]. If $\alpha(t)$ is the integral of a function $\phi(t)$, a corresponding theorem follows by the same methods. The conclusion can be obtained formally by differentiating both sides of equation (2) with respect to t. Using the more usual form of the Laplace transform, we state it as follows:

THEOREM B. *If $\phi(t)$ is continuous and such that the integral*

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt \tag{3}$$

converges for some value of s, then

$$\phi(t) = \lim_{s\to+\infty} s \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!}\, e^{kst} f(ks), \qquad (0 < t < \infty). \tag{4}$$

This inversion is of particular interest since it involves only the values of $f(s)$ itself (and not of its derivatives) for large positive values of s. It thus appears, at first sight, to be very different from the well known inversion [3, 288]

$$\phi(t) = \lim_{k\to+\infty} \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}, \qquad (0 < t < \infty). \tag{5}$$

The present author has shown [4] that the inversion operator on the right-hand side of equation (5) may be regarded symbolically, after an exponential change of variable, as $1/\Gamma(D)$, where $\Gamma(s)$ is the classical gamma function and D is the operation of differentiation. It is the purpose of the present note to show that the inversion of Phragmén and Fraser (4) is also symbolically $1/\Gamma(D)$, so that from this point of view formulas (4) and (5) are equivalent.

The exponential change of variable referred to above transforms equation (3) into

$$f(e^{-s}) = F(s) = \int_{-\infty}^{\infty} G(s-t)\Phi(t)dt,$$

$$G(t) = e^{-e^{-t}}, \qquad \Phi(t) = \phi(e^t)e^t.$$

Equation (4) becomes

$$\lim_{s\to+\infty} se^t \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{kse^t} F\left(\log\frac{1}{ks}\right) = \Phi(t). \tag{6}$$

We should like to show that equation (6) is equivalent, in some sense, to

$$\frac{1}{\Gamma(D)} F(t) = \Phi(t).$$

If we set $x = se^t$, equation (6) becomes

$$\lim_{x\to+\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} e^{kx} F(t - \log kx) = \Phi(t). \tag{7}$$

As is customary in operational theory, we define $e^{aD}F(t)$ to be $F(t+a)$, so that equation (7) may be written as

$$\lim_{x\to+\infty} \left[x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^D} \right] F(t) = \Phi(t).$$

Hence our result will be established if we can show that

$$\lim_{x\to+\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^y} = \frac{1}{\Gamma(y)}, \qquad (0 < y < \infty).$$

But this follows from Theorem B itself if we observe that

$$\frac{1}{s^y} = \int_0^{\infty} e^{-st} \frac{t^{y-1}}{\Gamma(y)} dt, \qquad (0 < y, s < \infty),$$

and take $\phi(t) = t^{y-1}/\Gamma(y)$, $f(s) = 1/s^y$, $t = 1$. Then Theorem B yields

$$\phi(1) = \frac{1}{\Gamma(y)} = \lim_{x \to +\infty} x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} \frac{e^{kx}}{(kx)^y},$$

and our assertion is proved.

References

1. W. C. G. Fraser, An inversion formula for an integral related to Dirichlet series, this MONTHLY, vol. 54 (1947), pp. 586–588.
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THE nTH DERIVATIVE OF A FRACTIONAL FUNCTION

V. F. IVANOFF, San Francisco

Let f be a fractional function,

$$f = \frac{g}{h}.$$

Then

$$\begin{aligned} hf &= g \\ h'f + hf' &= g' \\ h''f + 2h'f' + hf'' &= g'' \\ &\cdots\cdots\cdots\cdots \\ h^{(n)}f + \binom{n}{1} h^{(n-1)}f' + \binom{n}{2} h^{(n-2)}f'' + \cdots + hf^{(n)} &= g^{(n)}. \end{aligned}$$

Hence

$$f^{(n)} = \frac{1}{h^{n+1}} \begin{vmatrix} h & 0 & 0 & \cdots & 0 & g \\ h' & h & 0 & \cdots & 0 & g' \\ h'' & 2h' & h & \cdots & 0 & g'' \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h^{(n)} & \binom{n}{1} h^{(n-1)} & \binom{n}{2} h^{(n-2)} & \cdots & \binom{n}{1} h' & g^{(n)} \end{vmatrix}.$$

Incidentally, the solution of Problem E 789, this MONTHLY, vol. 54, 1947, p. 471; vol. 55, 1948, pp. 366–367, can be derived immediately from the above formula.

NOTE ON A THEOREM IN n-DIMENSIONAL GEOMETRY

O. TAUSSKY, Institute for Numerical Analysis, National Bureau of Standards, and
L. A. WIGGLESWORTH, University College, London

The following result due to Schäfli* turns up in various connections.

THEOREM 1. *If Δ_s is the determinant of the system of linear homogeneous equations*

$$(x_{i1}U_1 + \cdots + x_{in}U_n)(x_{j1}U_1 + \cdots + x_{jn}U_n) = 0, \qquad (i, j = 1, \cdots, n),$$

with the $\frac{1}{2}n(n+1)$ quantities U_kU_l as unknowns, $(k, l=1, \cdots, n)$, then $\Delta_s \neq 0$ if $\Delta = |x_{ik}| \neq 0$.

The above result can be interpreted by means of projective geometry. For it expresses the fact that there cannot exist a quadric which has a given non-degenerate simplex both inscribed in it and self polar with respect to it; the tangent space at each vertex would have to contain all the remaining vertices.

We now give a proof of Theorem 1 by means of metrical geometry. The proof depends on the generalization of a very well known theorem in elementary geometry: *Two triangles whose sides are of equal length are congruent.* The n-dimensional analogue can be stated in the following way:

THEOREM 2. *Let $P_i = (x_{i1}, \cdots, x_{in})$, $(i=0, \cdots, n)$, be $n+1$ (not necessarily real) points where $x_{01} = x_{02} = \cdots = x_{0n} = 0$ and*

$$\Delta = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \neq 0.$$

Then a transformation $x_k' = \sum_j a_{kj}x_j$ which carries these points into points (say P_i') with the same squared distances must necessarily be orthogonal:

$$\sum_k a_{kj}a_{kl} = \delta_{jl}.$$

Proof of Theorem 2. Since

$$x_{ik}' = \sum_j a_{kj}x_{ij}, \qquad (i, k, j = 1, \cdots, n),$$

it follows that

$$a_{kj} = \frac{\sum_i x_{ik}'X_{ij}}{\Delta}, \tag{1}$$

where X_{ij} is the cofactor of x_{ij} in Δ. Since

$$(P_iP_j)^2 = (P_i'P_j')^2, \qquad (i, j = 0, \cdots, n),$$

it follows that

* Denkschriften d. Akad. d. Wiss. Wien (2), 1857, pp. 1–24; actually, Δ_s is the $n(n-1)/2$th power of Δ.

$$\sum_k x_{ik}x_{jk} = \sum_k x'_{ik}x'_{jk}, \qquad (i, j = 1, \cdots, n). \tag{2}$$

From (1) and (2) we obtain

$$\sum_k a_{kj}a_{kl} = \frac{\sum_{i,k,h} x'_{ik}X_{ij}x'_{hk}X_{hl}}{\Delta^2} = \frac{\sum_{i,k,h} x_{ik}x_{hk}X_{ij}X_{hl}}{\Delta^2} = \sum_k \delta_{kj}\delta_{kl} = \delta_{jl},$$

which is the required result.

Proof of Theorem 1. Assume that $\Delta_s = 0$. Let ω_{lm}, $(l, m = 1, \cdots, n)$, be a non-trivial solution of the system of linear homogeneous equations with the quantities $\sum_k a_{kl}a_{km} - \delta_{lm}$ as unknowns:

$$\sum_l \Big(\sum_k a_{kl}^2 - 1 \Big)(x_{il} - x_{jl})^2 + \sum_{l<m} \sum_k 2a_{kl}a_{km}(x_{il} - x_{jl})(x_{im} - x_{jm}) = 0.$$

A non-trivial solution exists since the determinant Δ^* of this system coincides with Δ_s apart from a factor $(-2)^{n(n-1)/2}$. The rows

$$R_{ii} = (x_{i1}^2, x_{i2}^2, \cdots, x_{in}^2, 2x_{i1}x_{i2}, \cdots)$$

are common to both determinants. The row

$$R_{ij} = (x_{i1}x_{j1}, x_{i2}x_{j2}, \cdots, x_{in}x_{jn}, x_{i1}x_{j2} + x_{i2}x_{j1}, \cdots)$$

of Δ_s can be obtained from the row

$$((x_{i1} - x_{j1})^2, \cdots, (x_{in} - x_{jn})^2, 2(x_{i1} - x_{j1})(x_{i2} - x_{j2}), \cdots)$$

of Δ^* by adding $-R_{ii} - R_{jj}$ and multiplying by $-\frac{1}{2}$.

We can then find numbers a_{ik} such that

$$\sum_k a_{kl}a_{km} - \delta_{lm} = \omega_{lm}. \tag{3}$$

This is possible since it means to find n (not necessarily real) vectors in n dimensions $(a_{i1}, \cdots, a_{in})$ whose scalar products are given. This is equivalent to the problem of finding n (not necessarily real) points in n dimensions if their squared distances from $(0, \cdots, 0)$ and from each other are given. The latter is always possible.†

Consider then the linear transformation $x_k' = \sum_j a_{kj}x_j$ and its effect on the points P_i. Suppose that the transform of P_i is $P_i' = (x_{i1}', \cdots, x_{in}')$. Then

$$(P_i'P_j')^2 - (P_iP_j)^2 = \sum_l \Big(\sum_k a_{kl}^2 - 1 \Big)(x_{il} - x_{jl})^2 + \sum_{l<m} \sum_k 2a_{kl}a_{km}(x_{il} - x_{jl})(x'_{im} - x'_{jm}) = 0$$

† See A. Wald, Erg. Math. Koll (Vienna) 5, 1933, pp. 32–42, and O. Taussky, Erg. Math. Koll. (Vienna) 6, 1935, pp. 20–23. See also L. M. Blumenthal, Distance Geometries, University of Missouri Studies, (13), 1938, pp. 70–71.

because of (3). This is a contradiction because from Theorem 2 the transformation $x \to x'$ is orthogonal so that $\omega_{lm} = 0$.

Conversely, Theorem 2 is a consequence of Theorem 1. This can be shown by a mere rearrangement of the proof given for Theorem 1.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Institute for Advanced Study, Princeton, New Jersey.

TEACHING NOTE ON REPEATED ROOTS IN LINEAR DIFFERNTIAL EQUATIONS

J. B. REYNOLDS, Lehigh University

The following simple and direct method of proving the rule for double roots in the solution of linear differential equations with constant coefficients is seldom seen in elementary texts on the subject.

If

$$Ay^{(n)} + By^{(n-1)} + \cdots + Ly^{(2)} + My^{(1)} + Ny = 0 \tag{1}$$

is a differential equation in which $y^{(n)}$ stands for the nth derivative of y with respect to x and

$$f(m) = Am^n + Bm^{n-1} + \cdots + Lm^2 + Mn + N = 0, \tag{2}$$

and we substitute

$$y = c_1 e^{mx} + c_2 x e^{mx} \tag{3}$$

into (1) we find

$$[c_1 f(m) + c_2 f^1(m) + c_2 x f(m)]e^{mx} = 0 \tag{4}$$

in which $f^1(m)$ is the derivative of $f(m)$ with respect to m.

Equation (4) is satisfied, making (3) a solution of (1), if $f(m) = 0$ and $f^1(m) = 0$, simultaneously.

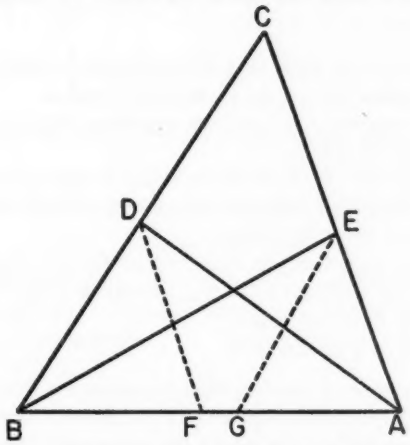
Since these are the conditions for (2) to have two equal roots, (3) is a solution of (1) in case (2) has two equal roots.

The method can readily be extended for the case of any number of multiple roots.

ANGLE BISECTORS OF AN ISOSCELES TRIANGLE

W. E. Bleick, U. S. Naval Postgraduate School

The following is an indirect proof that a triangle ABC is isosceles if the angle bisectors AD and BE are equal. If the triangle is not isosceles, one of the bisected angles must be greater than the other. Assume that angle EAB is greater than angle ABD. Construct a line through E parallel to BC and intersecting AB at G. Construct a line through D parallel to AC and intersecting AB at F. Then $EG>EA$ since $BC>AC$ by hypothesis. Also $DF>EG$ from triangles DFA and EGB. Now $AG=AB-EG$ and $BF=AB-DF$. Hence $BF<AG$ since $DF>EG$. The triangles ABC, AGE and FBD are similar. Hence



$BF/AG=DF/EA$, from which it follows that $DF<EA$ since $BF<AG$. But $EA<EG$ from above. Hence $DF<EG$ which contradicts the inequality $DF>EG$ above. Hence the bisected angles must be equal. This proof was obtained by Maria Goeppert Mayer in 1932.

EQUATION OF VALUE IN INVESTMENT THEORY

G. F. Rose, University of Wisconsin

Let us define an *investment system* to be any set of parties who have agreed upon a method of calculating the obligation of each one of them to each of the others.

An investment system is said to be in *equilibrium* if none of its parties has an obligation to any of the others. The *liquidation value* at a given time of any party A of an investment system is defined to be the net amount (with proper algebraic sign) that would be received by A in the process of instantaneously bringing the system into equilibrium at that time.

Any problem in the realm commonly denoted by "Mathematics of Finance"

can be reduced to the analysis of one or more investment systems. The almost universally used method of calculating obligations makes the mathematics of this analysis especially simple. Thus, it turns out that the liquidation value of any party A varies according to the following rules:

1. Upon completion of a payment to or from A, A's liquidation value is decreased or increased respectively by the amount of the payment.

2. Between payments, A's liquidation value varies at a fixed compound interest rate which will be referred to as A's *yield rate.*

The direct application of rules 1 and 2 constitutes the form of computation known as the *amortization table,* which consists of a payment-to-payment study of liquidation value. A more convenient procedure, however, is formulated by the following theorem.

THE EQUATION OF VALUE THEOREM: *The following process is valid for any party A whose liquidation value follows the above rules* 1 *and* 2.

1. *Consider the investment system over any time interval.*
2. *Choose any reference date.*
3. *Using A's yield rate, bring to the reference date A's liquidation value at the beginning of the interval and also all items leaving A's hands within the given interval. Sum the resulting values.*
4. *Using A's yield rate, bring to the reference date A's liquidation value at the end of the interval and also all items entering A's hands within the given interval. Sum the resulting values.*
5. *Equate the sum in* 3 *to that in* 4.

I wish to call attention to the fact that the equation of value process has not the status of an axiom; on the contrary, it stands on the prior concept of the amortization table procedure. To prove the equation of value theorem, let us consider a party A, with yield rate (j, m), over a time interval t_0 to t_{n+1}. Suppose that, within this interval, A makes and receives a total of n payments P_i; call their respective dates t_i $(t_0 < t_1 \leqq \cdots \leqq t_n < t_{n+1})$. Let $V(t)$ denote A's liquidation value at date t. Let r be any reference date. With rules 1 and 2 as our sole assumptions, we must establish the validity of

$$(1) \quad V(t_0)k^{r-t_0} + \sum_{(1 \leqq i \leqq n, P_i \text{ from } A)} P_i k^{r-t_i} = V(t_{n+1})k^{r-t_{n+1}} + \sum_{(1 \leqq i \leqq n, P_i \text{ to } A)} P_i k^{r-t_i},$$

where $k = (1+j/m)^m$.

Let I_i be P_i or $-P_i$ according as P_i is from A or to A respectively. We will prove by induction on i that A's liquidation value after payment at date $t_i (1 \leqq i \leqq n)$ is

$$(2) \quad V(t_i) = V(t_0)k^{t_i-t_0} + \sum_{j=1}^{i} I_j k^{t_i-t_j}.$$

In the first place $V(t_1) = V(t_0)k^{t_1-t_0} + I_1$. Furthermore,

$$V(t_{i+1}) = V(t_i)k^{t_{i+1}-t_i} + I_{i+1}$$

$$= \text{(by induction hypothesis)}\left[V(t_0)k^{t_i-t_0} + \sum_{j=1}^{i} I_j k^{t_i-t_j}\right]k^{t_{i+1}-t_i} + I_{i+1}$$

$$= V(t_0)k^{t_{i+1}-t_0} + \sum_{j=1}^{i+1} I_j k^{t_{i+1}-t_j}.$$

Hence (2) is established. As a consequence,

$$V(t_{n+1}) = V(t_n)k^{t_{n+1}-t_n}$$

$$= V(t_0)k^{t_{n+1}-t_0} + \sum_{j=1}^{n} I_j k^{t_{n+1}-t_j}$$

$$= V(t_0)k^{t_{n+1}-t_0} + \sum_{(1\leq j\leq n, P_j \text{ from } A)} P_j k^{t_{n+1}-t_j} - \sum_{(1\leq j\leq n, P_j \text{ to } A)} P_j k^{t_{n+1}-t_j},$$

from which (1) follows after multiplying through by $k^{r-t_{n+1}}$.

TANGENT TO A CIRCLE FROM AN EXTERIOR POINT

F. H. YOUNG, Oregon State College

The following method of finding the equation of a line tangent to a circle and passing through a point outside the circle has proved popular with my class in elementary analytic geometry.

Consider the circle

$$(x - h)^2 + (y - k)^2 = r^2$$

and the exterior point (x_1, y_1). Denote by S the length of tangent to the circle from (x_1, y_1). Denote by m_c the slope of the line joining (h, k) and (x_1, y_1). Denote by t the tangent of the angle between the line from (x_1, y_1) to (h, k) and the tangent line. Obviously, $t=r/S$. Then, by using the formula for $\tan(a \pm b)$, we obtain

$$m = \frac{m_c \pm t}{1 \pm m_c t},$$

the slopes of the tangent lines. Knowing the point and the slopes, we can easily obtain the desired equations.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 831. *Proposed by K. W. Crain, Purdue University*

If squares be constructed on the legs of a right triangle, the lines (which do not lie along the sides of the triangle) drawn from each end of the hypotenuse to a vertex of the opposite square intersect on the altitude which passes through the vertex of the right angle.

E 832. *Proposed by V. E. Dietrich, Purdue University*

If a circle has a center with at least one irrational coördinate, then there are at most two points on the circle with rational coördinates.

E 833. *Proposed by P. D. Thomas, Washington, D. C.*

A surveyor in laying out a square park area in a city found it was necessary because of obstructions to shorten two opposite sides by one foot, but both the length and width were in integral feet. To check his work he ran a diagonal of the resulting rectangle. Imagine his surprise to find that the semi-perimeter (diagonal+length+width) was in integral rods! What were the dimensions of the field?

E 834. *Proposed by Don Walter, Pomona College*

Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 \cdots \\ 0 & 1 & 1 & 0 & 1 & 0 \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix},$$

where F_n is the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, x, y, $x+y$, $\cdots$ and the determinant is of order $n-1$.

E 835. *Proposed by Kenneth May, Carleton College*

If x_1, y_1, and a are real numbers, and for all integral $n \geqq 1$ we have $x_{n+1} = a(x_n^2 - y_n^2)$ and $y_{n+1} = 2ax_ny_n$, for what values of x_1 and y_1 will $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$?

SOLUTIONS

Binomial Coefficients

E 794 [1947, 545]. *Proposed by Huan-Ting Kuo, National Wuhan University, China*

Show that

$$\begin{vmatrix} \binom{r+1}{r} & \binom{r+1}{r+1} & 0 & 0 & \cdots & 0 \\ \binom{r+2}{r} & \binom{r+2}{r+1} & \binom{r+2}{r+2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \binom{n}{r+3} & \cdots & \binom{n}{n-1} \end{vmatrix} = \binom{n}{r}.$$

Solution by R. A. Bradley, University of North Carolina. Let us give a proof by induction. We first note that the identity holds for $n=r+1$ and $r+2$. On the assumption that the identity is true for $n=r+1, r+2, \cdots, k-1$ we now show that it is true for $n=k$. Denote the determinant for $n=k$ by D_k. Expanding D_k repeatedly with respect to the elements of the last column we obtain

$$D_k = k\binom{k-1}{r} - \frac{k(k-1)}{2!}\binom{k-2}{r} + \frac{k(k-1)(k-2)}{3!}\binom{k-3}{r} - \cdots \pm \frac{k(k-1)\cdots(r+2)}{(k-r-1)!}\binom{r+1}{r} \mp \binom{k}{r}.$$

Factoring out $\binom{k}{r}(k-r)$ from all terms, $(k-r-1)/2$ from all but the first, $(k-r-2)/3$ from all but the first two, etc., we obtain

$$D_k = \binom{k}{r}(k-r)\left\{1 - \frac{k-r-1}{2}\left\{1 - \frac{k-r-2}{3}\left\{1 - \frac{k-r-3}{4}\left\{\cdots \left\{1 - \frac{2}{k-r-1}\left\{1 - \frac{1}{k-r}\right\}\right\}\cdots\right\}\right\}\right\}\right\}.$$

Successively simplifying the last bracket, the jth bracket is seen to be $(k-r-j)/(k-r)$, and continuing

$$D_k = \binom{k}{r}(k-r)\left\{1 - \frac{k-r-1}{2}\,\frac{2}{k-r}\right\} = \binom{k}{r}.$$

The identity is now established for all integral $n \geqq r+1$.

Also solved by J. C. Eaves, Shai-shi Ting, and John Williamson. Irving Kaplansky pointed out that the identity is a special case of a formula on p. 257 of Netto's *Lehrbuch der Combinatorik*. Reference is made there to Zeipel, Lunds Universitets Års-Skrift, 1865.

Tetrahedron and Perspective Triangles

E 795 [1947, 546]. *Proposed by N. A. Court, University of Oklahoma*

The pairs of points U', U''; V', V''; W', W'' are marked, respectively, on the edges DA, DB, DC of a tetrahedron $ABCD$. Determine three points U, V, W on the edges BC, CA, AB, respectively, so that the three lines joining the vertices of each of the triangles $UV'W''$, $VW'U''$, $WU'V''$, UVW to the corresponding vertices of the triangles DCB, DAC, DBA, ABC, respectively, shall have a point in common.

Solution by the Proposer. If triangle $UV'W''$ is perspective to DCB, the line $V'W''$ meets BC in the harmonic conjugate P of U with respect to the points B, C. Similarly, the lines $W'U''$, $U'V''$ meet the edges CA, AB in the harmonic conjugates Q, R of V, W with respect to the pairs of points C, A; A, B. Now if triangle UVW is perspective to ABC, the three points P, Q, R are collinear, for they belong to the axis of perspectivity of ABC and UVW.

Thus a necessary condition for the problem to have a solution is that the three points $P=(V'W'', BC)$, $Q=(W'U'', CA)$, $R=(U'V'', AB)$ shall be collinear. It is readily seen that this condition is also sufficient. When this condition is fulfilled the problem has only one solution.

Coin-Tossing Experiment

E 796 [1946, 596]. *Proposed by Henry Scheffé, University of California at Los Angeles*

Describe a coin-tossing experiment in which the probability of success is one-third.

I. *Solution by R. V. Andree, University of Wisconsin.* Following are some solutions:

1. Five uniform coins are tossed until they do *not* all show the same face. Then (a) the probability of exactly two heads is 1/3, (b) the probability of exactly three heads is 1/3, (c) the probability of a four to one split is 1/3.

2. Given one normal coin and one two headed coin, one coin is selected at random and tossed. It comes down heads. Then the probability that it is the normal coin is 1/3.

3. A game of chance is played as follows: A has two coins, B has one coin. All three coins are tossed. If all three coins agree, each player takes his own coins back and no exchange is made. Otherwise A wins if both of A's coins disagree with that of B and B wins if only one of A's coins disagrees with that of B. Then the probability that A will win is 1/3.

4. An interesting possibility is brought up by discarding the assumption that the coin is "thin" and thus that the probability of its standing on edge is negligible. Let us consider a coin the ratio of whose diameter to height is $\sqrt{3}/1$ Then the probability that this coin will land heads is 1/3.

II. *Solution by Leo Moser, University of Manitoba.* Following are some solutions:

1. Two coins are tossed repeatedly until not both of them fall tails. If both of them are then heads the experiment is counted a success. The probability of success is 1/3.

2. A single coin is tossed repeatedly until the number of heads is one greater than, or two less than, the number of tails. The probability of the latter case is 1/3. This method can be used to obtain an experiment having any preassigned *rational* number as probability of success. For proof of this see Whitworth's *Choice and Chance*, p. 231.

3. We describe a coin-tossing experiment E_p having probability of success p, where p is any preassigned *real* number, $0 \leqq p \leqq 1$. Express p as a binary decimal, $p = \sum_{i=1}^{\infty} a_i/2^i$, $a_i = 0$ or 1. Where two expressions are possible, as for $\frac{1}{2} = .1\dot{0} = .0\dot{1}$, take either one. Now toss a coin repeatedly and let $b_i = 1$ if the ith toss results in heads and 0 if the ith toss results in tails. Let the experiment terminate when first $a_i \neq b_i$ and be a success if then $a_i > b_i$. The probability that the experiment will never terminate is clearly 0. The probability that it will terminate in success at the ith toss is $a_i/2^i$, so that the total probability of success is p.

III. *Solution by A. C. Cohen, Jr., University of Georgia.* The following simple case of the "king bee" matching scheme furnishes a solution. Three individuals agree to match, each tossing one coin. They agree to toss until one member of the trio is "odd," and the "odd" man is declared winner. The probability that any designated member of the trio be declared winner is 1/3.

IV. *Solution by J. P. Kelly, Carbide and Carbon Chemicals Corp., Oak Ridge, Tennessee.* A and B toss a coin alternately until one gets a head. If A tosses first, then the probability of B getting a head first is 1/3.

Also solved by Colin Blyth, G. F. Cramer, George Grossman, P. R. Hill, Sr., H. G. Landau, C. F. Pinzka, Hanan Rubin, E. D. Schell, R. J. Walker, and the proposer.

Ellipses in a Circle

E 797 [1947, 596]. *Proposed by C. O. Hines, University of Toronto*

If ellipses are described on diameters of a given circle as major axis, and such that they all pass through a given point (within, or on the boundary of, the circle), then they will also all pass through a second point, symmetrical about the center to the first, and the locus of their foci will be an ellipse having the two fixed points as foci and the common diameter as major axis.

Solution by Colin Blyth, University of North Carolina. Let A and B, the symmetrical fixed points, lie on the diameter MON of the circle, whose center is O. Let C and D be the foci of a general ellipse of the given family. Then $CA + AD = MN$. But, by symmetry, $CB = AD$. Therefore $CA + CB = MN$, and C lies on

the ellipse with foci A and B and major axis MN.

Also solved by R. V. Andree, W. G. Brady, Ragnar Dybvik, Bradford Hadnot, Roger Lessard, Miriam Millman, Margaret Olmsted, P. D. Thomas, Maud Willey, and the proposer.

Distribution of Suits in a Bridge Hand

E 798 [1948, 30]. *Proposed by J. M. Elkin, Chicago, Ill.*

Prove that $\Pi(n_i!)$, where Σn_i is constant, is a minimum when $\Sigma|n_i-n_j|$ is a minimum, and that, consequently, the most likely distribution of the four suits in a bridge hand is four cards of one suit and three cards each of the other three suits.

Solution by Leo Moser, University of Manitoba. If $n_i>n_j+1$ then

$$n_i!n_j! = (n_i - 1)!(n_j + 1)!n_i/(n_j + 1) > (n_i - 1)!(n_j + 1)!.$$

Hence a necessary and sufficient condition that $\Pi(n_i!)$ be minimum with Σn_i constant is that $|n_i-n_j| \leqq 1$. But, since we are dealing with integers, a necessary and sufficient condition that $|n_i-n_j| \leqq 1$ with Σn_i constant is that $\Sigma|n_i-n_j|$ be a minimum. This completes the first part of the problem.

The distribution 4, 3, 3, 3, however, is not the most probable distribution of suits in a bridge hand. The actual state of affairs and also the 'common mistake' leading to the statement made in the problem are discussed in *Mathematical Puzzles for Beginners and Enthusiasts*, Geoffrey Mott-Smith, pp. 91, 101, 214.

Also solved, partially, by Richard Courter, E. D. Schell, and the proposer.

Editorial Note. It would seem that the theorem of the problem cannot be directly applied to the problem of finding the most likely distribution of the four suits in a bridge hand. The number of bridge hands with n_1 cards of one suit, n_2 of another, n_3 of another, and n_4 of another, is

$$4C(13, n_1)C(13, n_2)C(13, n_3)C(13, n_4) = 4(13!)^4/\Pi[n_i!(13 - n_i)!],$$

where $n_1+n_2+n_3+n_4=13$. By actual computation this is found to be a maximum for the distribution 4, 4, 3, 2. This distribution, then, is the most likely. It occurs better than once in five hands, and is over twice as frequent as the 4, 3, 3, 3 distribution incorrectly supposed by the proposer to be the most frequent. For the numbers of bridge hands corresponding to the various suit distributions see *Bridge Hands*, this MONTHLY [1941, 329].

A Combinatorial Identity

E 799 [1948, 30]. *Proposed by Leo Moser, University of Manitoba*

Prove that for all n

$$\sum_{r=0}^{n}\left[2^{n-r}\binom{n}{r}^2 - \binom{2n-r}{n}\binom{n}{r}\right] = 0.$$

I. *Solution by J. G. Herriot, Stanford University.* We have

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}$$

and

$$(2+x)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} x^r.$$

It is obvious that

$$\sum_{r=0}^{n} 2^{n-r} \binom{n}{r}^2$$

is the coefficient of x^n in the expansion of $(x+1)^n(2+x)^n$. But

$$(x+1)^n(2+x)^n = (1+x)^n[(1+x)+1]^n = \sum_{r=0}^{n} \binom{n}{r} (1+x)^{2n-r}.$$

The coefficient of x^n in the last expression is

$$\sum_{r=0}^{n} \binom{2n-r}{n} \binom{n}{r}.$$

Hence

$$\sum_{r=0}^{n} 2^{n-r} \binom{n}{r}^2 = \sum_{r=0}^{n} \binom{2n-r}{n} \binom{n}{r},$$

which is equivalent to the required expression.

II. *Solution by E. D. Rainville, University of Michigan.* Let $P_n(x)$ be the standard Legendre polynomial. We shall show that the left member of the equation in the problem is $[P_n(3) - P_n(3)]$, and therefore vanishes for all n.

Two standard forms for the Legendre polynomial are

$$P_n(x) = \sum_{r=0}^{n} \binom{n}{r}^2 \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \tag{1}$$

and

$$P_n(x) = F\left(-n,\ n+1,\ 1,\ \frac{1-x}{2}\right) = \sum_{r=0}^{n} \frac{(n+r)!}{(r!)^2(n-r)!} \left(\frac{x-1}{2}\right)^r. \tag{2}$$

Replacing r by $(n-r)$ and reversing the order of summation in (2) yields

$$P_n(x) = \sum_{r=0}^{n} \frac{(2n-r)!}{r![(n-r)!]^2}\left(\frac{x-1}{2}\right)^r$$

or

$$P_n(x) = \sum_{r=0}^{n}\binom{2n-r}{n}\binom{n}{r}\left(\frac{x-1}{2}\right)^r. \tag{3}$$

Equations (1) and (3) lead us to the identity, for integral n and any x,

$$\sum_{r=0}^{n}\binom{n}{r}\left[\left(\frac{x+1}{2}\right)^{n-r}\binom{n}{r} - \binom{2n-r}{n}\right]\left(\frac{x-1}{2}\right)^r = 0. \tag{4}$$

Put $x=3$ in (4) to get the proposer's identity.

Also solved by E. D. Schell, F. C. Smith, and F. B. Thompson. The proposer stated that he obtained the identity by solving in two entirely different ways the following problem: What is the number of "paths" from (0, 0) to (n, n), the possible "moves" being of three kinds—(1) increasing the x coördinate by 1, (2) increasing the y coördinate by 1, and (3) increasing both the x and y coördinates by 1?

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4310. *Proposed by Paul Erdös, Syracuse University*

Let $f(z)=z^n+\cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)|\leqq 1$. Prove that there always exists a z_0 in A_f with $|f'(z_0)|\geqq n$. Equality occurs only for z^n.

4311. *Proposed by V. L. Klee, Jr., University of Virginia*

If k and x are positive integers, let $f_k^1(x)=k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j=2, 3, \cdots$, let

$$f_k^j(x) = f_k^{j-1}[f_k^1(x)].$$

Show that for $k \leqq 3$, the sequence, $f_k^1(x), f_k^2(x), \cdots$, is eventually constant, while for $k \geqq 4$, the sequence is eventually monotonically increasing.

4312. *Proposed by R. C. Lyness, Staffordshire, England*

In order to help our school savings campaign, Morris organized a lottery. Certain members of the savings group were each issued a book containing a gross of tickets. The tickets were sold for a penny each and the prize for the winning ticket was a number of sixpenny savings stamps. When I asked Morris how many stamps the winner got, he said that I could work it out for myself. All he need tell me was that before the winning ticket was drawn all the ticket sellers had assembled around a circular table and each had handed in his takings and his unsold tickets; that (a) no two had sold the same number; that (b) the number of unsold tickets returned by each seller was in every case equal to the product of the values in shillings of the tickets sold by his two neighbors at the table; and that (c) when the ticket sellers were rewarded with a penny each for their trouble the sum left over exactly bought the winner's stamps.

How many sellers were there and how many tickets did each sell?

4313. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Two perpendicular chords AB, CD of a circle O intersect at a point P inside the circle. There are eight circles O_1, O_2, O_3, O_4, O_1', O_2', O_3', O_4' tangent to both chords and also to the circle O, the first four exteriorly, the last four interiorly and such that O_i and O_i' lie in opposite quadrants formed by the given chords. (1) The sum of the squares of the distances O_1O_1', O_2O_2', O_3O_3', O_4O_4' is independent of the position of P, and the products $(O_1O_1')(O_3O_3')$ and $(O_2O_2')(O_4O_4')$ are equal. (2) The radical axes of the circles O_1, O_1', O_3, O_3' taken in pairs and those of the circles O_2, O_2' O_4, O_4' taken in pairs, intersect one another in thirty-six points of which twelve are on the circle O, four of these coinciding with the vertices of the square whose diagonals are parallel to AB and CD.

4314. *Proposed by N. J. Fine, Washington, D. C.*

Prove the identity:

$$1+\frac{x}{(1-x)^2}+\frac{x^2}{(1-x)^2(1-x^2)^2}+\frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\cdots = \frac{1-x+x^3-x^6+x^{10}-\cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2}.$$

SOLUTIONS

Circles and a Locus Problem

4165 [1945, 346] Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Let A and B be two fixed points of a given circle while C and D are two variable points of the same circle such that the arc length CD remains constant.

The orthogonal projection of D on AC is P and on BC is Q. Show that (1) The Simson line of C and D for the triangles ABD and ABC have fixed directions. (2) The centers of the circles tritangent to DPQ (inscribed and escribed) describe two Pascal limaçons.

*Solution by the Proposer.** Suppose arcs AB and CD are each less than a semi-circumference and have the same sense.

1. Let B', A' be the points where the perpendiculars DP, DQ to the sides AC and BC of triangle ABC again cut the given circle (O); let O', O_1, O_2 be the midpoints of the chords CD, AB', $A'B$. The circle (O') on chord CD as diameter passes through P and Q; the arcs AB', $A'B$ are equal, each being equal to the supplement of arc CD; their chords are twice the length of OO'; the lines AA', BB', O_1O_2 are parallel; the segments O_1P, O_2Q are equal and parallel to OO'; O_1PQO_2 is a parallelogram and PQ is equal and parallel to O_1O_2. When chord CD moves so as to retain the same length, the points A', B', O_1, O_2 remain fixed; hence the direction of the Simson line PQ of point D for triangle ABC does not change.

2. The bisectors of the interior and exterior angles at D of triangle DPQ cut the circles (O') and (O) again at diametrically opposite points E, F of (O') and G, H of (O); EF is perpendicular to PQ and GH is perpendicular to $A'B'$. The lines CE, CF cut (O) again at K and L respectively. Therefore the chords GK and HL are each equal to twice the length OO'. When chord CD moves, the points G, H, K, L remain fixed and points E, F describe the circles (O_4) and (O_5) on diameters GK and HL respectively. Now the centers of the circles tritangent to triangle DPQ are the points where the two circles through P and Q with centers E and F cut the lines DG and DH respectively. The locus of these centers then are Pascal limaçons whose poles are G and H and whose director circles are (O_4) and (O_5) with parameters EP and FP for which

$$EP^2 + FP^2 = CD^2, \quad \text{and} \quad 2EP \cdot FP = CD \cdot O_1O_2.$$

A particular case arises when AB and CD are diameters of (O). The locus of centers of circles tritangent to triangle DPQ is then formed by circles through A and B with centers at E and F respectively.

Editorial Note. The translator adds the following remarks. Using isotropic coördinates, we may choose the circumcircle to be the given circle, and represent points A, B, C by unit vectors α, β and γ respectively. Let D be represented by the unit vector t^2; then, since the arc CD is constant in magnitude, we may replace γ by t_0t^2 where t_0 is constant with unit modulus. From these data we obtain $\alpha\beta t_0$ is the clinant of the Simson line of D for triangle ABC and $\alpha\beta/t_0$ as that of the Simson line of C for triangle ABD; since each of these is independent of t, the Simson lines have fixed directions. The map equation of the locus of the

* Translated from the French by O. J. Ramler, Catholic University of America.

incenter and excenter of triangle DPQ within the angle D is

$$2z = (1 + t_0)t^2 + (\alpha^{1/2} + \beta^{1/2})(1 - t_0)t + (t_0 - 1)\alpha^{1/2}\beta^{1/2}$$

which is a limaçon unless $1+t_0=0$. $t=\sqrt{\alpha}$ gives $z=\alpha$, and $t=\sqrt{\beta}$ gives $z=\beta$. Thus the locus goes through A and B. If $t_0=-1$, CD is a diameter, and the map equation reduces to that of a circle. Thus it appears that the locus degenerates to a circle when CD is a diameter, whether AB is a diameter or not.

Howard Eves notes that part (1) follows immediately from the theorem on p. 206 of Johnson's *Modern Geometry*. For, by this theorem, the Simson line of C for ABD must be perpendicular to BC', the isogonal of BC for angle ABD. But as C and D move along the circle, arc length CD remaining constant, BC' remains fixed. Similarly for the Simson line of D for ABC.

A Square with Curious Digit Pattern

4227 [1946, 594]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what number systems is the square of a number of three digits of the form bcb of the form $aabbcc$?

Partial solution by C. R. Phelps, Rutgers University. If r is the base of the number system, we have by hypothesis

$$(br^2 + cr + b)^2 = ar^5 + ar^4 + br^3 + br^2 + cr + c = (r + 1)(ar^4 + br^2 + c). \tag{1}$$

If $r+1$ has no square factor, then $br^2+cr+b\equiv 0 \bmod r+1$, whence $2b-c=k(r+1)$. Since $0<b<r$, $0\leqq c<r$, k is either 0 or 1. Suppose $k=0$, then $c=2b$ and (1) becomes

$$(br^2 + 2br + b)^2 = (r + 1)(ar^4 + br^2 + 2b),$$

hence

$$b^2(r + 1)^2 = ar^4 + br^2 + 2b. \tag{2}$$

Expanding, we get

$$ar^4 - b^2r^3 + (b - 3b^2)r^2 - 3b^2r - (b^2 - 2b) = 0. \tag{3}$$

The leading coefficient b of the number bcb cannot vanish. If b is not a divisor of a, since all coefficients in (3) after the first are divisible by b, Eisenstein's criterion implies that there is no rational root r unless b^2 divides the constant term b^2-2b. This gives $b=1, 2$. With $b=1$ in (3) we have $r=\pm 1$, which is impossible. With $b=2$, $c=4$ and (3) becomes $ar^3-4r^2-10r-12=0$. 6 and 12 are the only rational roots (>4) possible; $r=12$ gives nonintegral a; but $r=6$ gives $a=1$. Thus one solution is

$$242^2 = 112244 \quad \text{to the base 6.}$$

If in (3) b is a divisor of a, say $a=nb$, then (2) becomes

$$br^3+3br^2+3br+b=nr^4+r^2+2.$$

Hence, since the base r is a divisor of $b-2$, we must have $b-2=0$ and $r=6$ or 3; but this gives $n=1/2$ or $n=13/9$, both contrary to the assumption that b divides a.

The cases in which $k=1$ or in which $r+1$ has a square factor present complications. There are no instances, however, with $r<100$.

Editorial Note. The Proposer has carried considerably further the investigation of cases not included in Phelps' treatment above without, however, finding any further solutions or completing the proof that no further solutions exist.

Numbers Having Same Digits in Two Systems of Numeration

4240 [1947, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relations which must connect N, B, B', in order that the number N may be written with the same three digits in the system of numeration of base B as in the system ot base B'. Having given B, find B' and N. Apply the results when $B=10$.

Solution by G. W. Walker, Buffalo, New York. Let x', y', z' be the digits x, y, z in any order and suppose

$$xB^2+yB+z=x'B'^2+y'B'+z'. \tag{1}$$

For fixed values of B and B', this equation can be put in the form

$$px+qy+rz=0, \tag{2}$$

where p, q, r are integral functions of B and B'. Infinitely many sets of integral values of x, y, z exist which satisfy (2), viz.

$$x=k_3q/(p,q)-k_2r/(p,r), \qquad y=k_1r/(q,r)-k_3p/(p,q),$$
$$z=k_2p/(p,r)-k_1q/(q,r),$$

where (a, b) is the (positive) greatest common divisor of a and b, and k_1, k_2, k_3 are arbitrary integers. For our problem, we require in addition

$$0\leqq x,\quad y,\quad z<B,\quad B'. \tag{3}$$

Values satisfying both (2) and (3) need not exist, but if they exist they are easily found by trial.

If $B=10$, we take successive values for B' and seek solutions for x, y, z subject to (3). The method is essentially one of trial, although numerous ways of saving work appear in practice. All solutions thus far discovered for $B=10$ are listed below. Numbers for which the initial digit is zero to either base have been excluded.

$265_{10} = 526_7$	$774_{10} = 477_{13}$	$825_{10} = 258_{19}$
$316_{10} = 631_7$	$834_{10} = 438_{14}$	$551_{10} = 155_{21}$
$158_{10} = 185_9$	$261_{10} = 126_{15}$	$912_{10} = 219_{21}$
$227_{10} = 272_9$	$371_{10} = 173_{16}$	$511_{10} = 115_{22}$
$445_{10} = 544_9$	$913_{10} = 391_{16}$	$910_{10} = 190_{26}$
$196_{10} = 169_{11}$	$782_{10} = 278_{18}$	$911_{10} = 191_{26}$
$283_{10} = 238_{11}$	$441_{10} = 144_{19}$	$\cdots = \cdots$
$370_{10} = 307_{11}$	$518_{10} = 185_{19}$	$919_{10} = 199_{26}$
$191_{10} = 119_{13}$	$882_{10} = 288_{19}$	$961_{10} = 169_{28}$

Note in particular: $912_{10} = 219_{21} = 192_{26}$; $913_{10} = 391_{16} = 193_{26}$.
Also partially solved by Murray Barbour.

Editorial Note. The interest in such problems as this is not so much as studies in scales of notations as for the other questions which they suggest in elementary number theory. For example, suppose in the above problem that x, y, z, are given arbitrarily with x', y', z' an arbitrary permutation of x, y, z, and suppose that xx' is not a perfect square. There are then infinitely many sets of values of B and B' which satisfy (1) and (3). In fact, if p, q are positive integers for which

$$p^2 - xx'q^2 = 1,$$

it is easy to verify the B_{n+1}, B'_{n+1} as given below, will satisfy (1) whenever B_n, B'_n do:

$$\begin{aligned} B_{n+1} &= (p^2 + xx'q^2)B_n + 2x'pqB'_n + (pqy' + q^2x'y) \\ B'_{n+1} &= 2xpqB_n + (p^2 + xx'q^2)B'_n + (pqy + q^2xy'). \end{aligned} \tag{4}$$

Now $B_0 = B'_0 = 1$ satisfies (1). Therefore (4) gives an infinite sequence of sets of values of B, B' all of which, from a certain point on, will satisfy also (3). This particular sequence does not in general include all solutions.

Again, we may find in many ways parametric solutions of (1) after assigning a relation between B' and B. Thus if $B' = 2B+1$, $x' = y$, $y' = z$, $z' = x$, we have

$$B^2x + By + z = (2B + 1)^2y + (2B + 1)z + x,$$

whence at once $x+y \equiv 0 \bmod B$, so that $x+y = B$. Upon eliminating x and dividing by B we find

$$B^2 - 1 = (5B + 3)y + 2z.$$

Here $2z-2y\equiv 0 \bmod B+1$ or $2z-2y=m(B+1)$ with $m=0,\ 1,\ -1$. Thus we have $B-1=5y+m$. Finally

$$(m=0),\quad B=5y+1,\quad z=y,\quad x=4y+1,\quad \text{with } y \text{ arbitrary;}$$
$$(m=1),\ B=5y+2,\quad z=\tfrac{1}{2}(7y+3),\quad x=4y+2,\quad \text{with } y \text{ odd.}$$

$m=-1$ is excluded since it implies $z<0$.

A few of the many other sets which may be obtained similarly are:

$x=3y+3,\ z=3y+1,\ B=4y+3,\ B'=B+1;\ xyz_B=zxy_{B'};\ y$ arbitrary.

$x=4y-1,\ B=5y-2z-1,\ B'=2B-1;\ xyz_B=yzx_{B'};z,\ y(>2z)$ arbitrary.

$y=2x+z+1,\ B=3x+z+1,\ B'=B+1;\ xyz_B=yzx_{B'};\ x,\ z$ arbitrary.

$x=4y+1,\ B=7y+2,\ B'=2B;\ xyz_B=yxz_{B'};\ y,\ z(<B)$ arbitrary.

$x=z+2,\ y=0,\ B=2z+1,\ B'=B+2;\ xyz_B=zyx_{B'};\ z$ arbitrary.

$x=u^2,\ y=2uv,\ z=v^2,\ B=kv+1,\ B'=ku+1;\ xyz_B=zyx_{B'}$; with

$u,\ v,\ k$ arbitrary except that $u>v,\ kv>\max(u^2-1,\ 2uv-1)$.

A Tetrahedron of Maximum Volume

4243 [1947, 168]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The point M, situated in the interior of a tetrahedron $ABCD$, such that the volume of the tetrahedron having for vertices the points of intersection of the lines AM, BM, CM, DM, with the opposite faces of $ABCD$ be a maximum, coincides with the centroid of $ABCD$.

Solution by L. M. Kelly, University of Missouri. Let the barycentric coördinates of the point M be $x_1,\ x_2,\ x_3,\ x_4$, subject to the condition $x_1+x_2+x_3+x_4=1$. Then the coördinates of the points of intersection of AM, BM, CM, DM with the respective opposite faces will be $(0,\ x_2,\ x_3,\ x_4)$, $(x_1,\ 0,\ x_3,\ x_4)$, $(x_1,\ x_2,\ 0,\ x_4)$, $(x_1,\ x_2,\ x_3,\ 0)$. The volume of the tetrahedron in question will be

$$V=C\begin{vmatrix} 0 & x_2 & x_3 & x_4 \\ x_1 & 0 & x_3 & x_4 \\ x_1 & x_2 & 0 & x_4 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = Cx_1x_2x_3x_4\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix},$$

where C is a constant. It is thus a question of maximizing the product $x_1x_2x_3x_4$ subject to the condition that $x_1+x_2+x_3+x_4=1$. It is clear that this occurs when $x_1=x_2=x_3=x_4$ and hence when M is the centroid. It is equally clear that this theorem is true for a simplex of n dimensions.

Also solved by R. Bouvaist and the Proposer, who notes that the analogous problem for $n=3$ is solved in *Mathesis*, 1885, p. 87.

Perfect Squares of Special Form

4244 [1947, 232]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider squares of $2n$ digits which can be formed by bringing together two consecutive n-digit numbers. Show that in every system of numeration there is at least one unending sequence of such squares. Show also that, if n and B are given and if x^2 is such a square, then y^2 is another where $x+y=B^n+1$ (provided squares with initial zeros are admitted). As a particular example let $B=12$, $n=3$.

Solution by Roger Lessard, École Polytechnique, Montreal. Let A be an n-digit number in the scale of B and let

$$x^2 = AB^n + (A+1) = A(B^n+1) + 1, \qquad A+1 < B^n, \tag{1}$$

whence

$$(x+1)(x-1) = A(B^n+1). \tag{2}$$

Consider

$$x+1 = ab, \qquad x-1 = cd, \qquad A = ac, \qquad B^n+1 = bd,$$

and suppose that b and d are chosen relatively prime and having the product B^n+1. We require also $b, d>2$; otherwise we have $x\pm1 \geqq B^n+1$, whence $A+1 \geqq B^n$. Then $ab-dc=2$ has an infinite number of positive solutions:

$$a = a_1 + kd, \qquad c = c_1 + kb,$$

where a_1 and c_1 are the unique solutions for which $0<a_1<d$, $0<c_1<b$. Since $A=ac$ is greater than B^n if $k\neq0$, a_1 and c_1 provides the only valid solution.

We will therefore get as many solutions for given B and n as there are ways in which B^n+1 is separable into two relatively prime factors each of which exceeds 2. If B^n+1 is divisible by p distinct primes, the number of solutions will be 2^p-2, except however that if B^n+1 is even, 2 is not counted as one of the primes unless B^n+1 is divisible by 4. But B^n+1 is composite with at least two prime factors larger than 2 whenever n is not a power of 2. Therefore there are solutions for any base B and for most of the values of n.

If we substitute $x=B^n+1-y$ in (1) we have

$$(B^n+1-y)^2 - 1 = A(B^n+1), \tag{3}$$

from which we get

$$y^2 = A'B^n + A' + 1, \qquad A' + x = A + y. \tag{4}$$

Therefore y is another solution whenever x is one.

If $B=12$, $n=3$, we have $B^n+1=7\cdot13\cdot19$. Corresponding to $d=7$, 13, 19, $7\cdot13$, $7\cdot19$, $13\cdot19$ we use the above method to find that x^2 equals respectively $02E030$, 193194, 283284, 340341, 468469, $8T08T1$. These values of x^2 are written in the scale of 12 with T and E as the digits for ten and eleven.

Note that if we interchange the values of b and d we get the corresponding y instead of x.

Also solved by Murray Barbour and the Proposer.

Editorial Note. The proposed problem admits also the interpretation

$$x^2 = AB^n + (A - 1). \tag{1'}$$

For such values of x the argument in (3) and (4) above still holds, for we have $x+y=B^n+1$, $x-y=A-A'$, which imply

$$y^2 = A'B^n + A' - 1. \tag{4'}$$

There are no numbers of this form for $B=12$, $n=3$ because B^n+1 is a multiple of 7 and therefore cannot be a divisor of x^2+1.

The Proposer points out that for $x=B^m$, the number x^2 consists of the $2m$-digit number $A=00\cdots01$ followed by $2m$ zeros. Therefore for $y=B^{2m}+1-B^m$ we shall have $A'=y-x+A=B^{2m}-2B^m+2$. Thus y^2 is the $4m$-digit number

$$\overline{B-1}\ \overline{B-1}\cdots\overline{B-1}\ \overline{B-2}\ 0\ 0\cdots 2\ \overline{B-1}\ \overline{B-1}\cdots\overline{B-1}\ \overline{B-2}\ 0\ 0\cdots 0\ 1$$

in which there are $m-1$ digits in each set of $(B-1)$'s and in each set of zeros. We thus have an explicit form for the members of an infinite sequence of squares as required.

The Proposer suggests also the study of squares formed by the juxtaposition of A and $A+2$, or in general A and $A+a$. For example, with $B=10$, $9011^2=81198121$, $990^2=00980100$, $7312^2=53465344$, $2689^2=07230721$.

For squares of $2n$ digits formed by bringing together two equal n-digit numbers, see the National Mathematics Magazine, Problem no. 431, May 1942, p. 409.

nth Derivatives

4246 [1947, 232]. *Proposed by J. A. Greenwood*

Evidently $y'=x'Dy$, $y''=(x''D+x'^2D^2)y$, $\cdots$, where $D\equiv d/dx$ and primes indicate differentiation with respect to t. Find an explicit expression for $y^{(n)}$.

Solution by Robert Breusch, Amherst College. Consider the partitions of n into positive integers,

$$n = p_1 + 2p_2 + 3p_3 + \cdots + rp_r.$$

An expression for $y^{(n)}$ may be given as

$$y^{(n)} = \sum K(p_1, p_2, \cdots, p_r)x'^{p_1}x''^{p_2}\cdots x^{(r)p_r}D^{p_1+\cdots+p_r}y, \tag{1}$$

$$K(p_1, p_2, \cdots, p_r) = \frac{n!}{(2!)^{p_2}(3!)^{p_3}\cdots(r!)^{p_r}p_1!p_2!\cdots p_r!},$$

where the summation is taken over all possible partitions of n. The formula is correct for $n=1$ and $n=2$. Assume that it holds for any particular n and dif-

ferentiate. From any term of (1) we will obtain:

a) one term $x'^{p_1+1}x''^{p_2}\cdots x^{(r)p_r}D^{p_1+1+p_2+\cdots+p_r}y$, and

b) further terms in which one exponent decreases by unity and the following exponent increases by unity, while the remaining exponents and the index of D remain unchanged; as, for example,

$$x'^{p_1}x''^{p_2-1}x'''^{p_3+1}x''''^{p_4}\cdots x^{(r)p_r}D^{p_1+\cdots+p_r}y.$$

In each case the sum of the exponents of $x', x'', \cdots, x^{(r)}$ is equal to the index of D, while the sum of the weighted exponents is $n+1$. Thus $y^{(n+1)}$ is a sum of terms:

$$y^{(n+1)} = \sum C(q_1, q_2, \cdots, q_s)x'^{q_1}x''^{q_2}\cdots x^{(s)q_s}D^{q_1+\cdots+q_s}y, \tag{2}$$

$$q_1 + 2q_2 + 3q_3 + \cdots + sq_s = n + 1.$$

Now, considering the different ways in which

$$x'^{q_1}\cdots x^{(s)q_s}D^{q_1+\cdots+q_s}y$$

can be obtained through differentiation of terms in (1), we have

$$\begin{aligned} C(q_1, \cdots, q_s) = {} & \frac{n!}{(2!)^{q_2}\cdots(s!)^{q_s}(q_1-1)!q_2!\cdots q_s!} \\ & + \frac{n!(q_1+1)}{(2!)^{q_2-1}(3!)^{q_3}\cdots(s!)^{q_s}(q_1+1)!(q_2-1)!q_3!\cdots q_s!} \\ & + \frac{n!(q_2+1)}{(2!)^{q_2+1}(3!)^{q_3-1}\cdots(s!)^{q_s}q_1!(q_2+1)!(q_3-1)!\cdots q_s!} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + \frac{n!(q_{s-1}+1)}{(2!)^{q_2}\cdots\{(s-1)!\}^{q_{s-1}+1}(s!)^{q_s-1}q_1!\cdots(q_{s-1}+1)!(q_s-1)!}. \end{aligned}$$

Here, whenever $t=0$, $1/(t-1)!$ is to be understood as having the value 0, so that $1/(t-1)!=t/t!$ is generally correct. Then finally

$$\begin{aligned} C(q_1, \cdots, q_s) &= \frac{n!(q_1+2q_2+3q_3+\cdots+sq_s)}{(2!)^{q_2}(3!)^{q_3}\cdots(s!)^{q_s}q_1!q_2!\cdots q_s!} \\ &= \frac{(n+1)!}{(2!)^{q_2}\cdots(s!)^{q_s}q_1!\cdots q_s!}, \end{aligned}$$

and the proof is complete by induction. Also solved by Chih-yi Wang.

Isosceles n-Points

4254 [1947, 345]. *Proposed by Paul Erdös, Syracuse University*

We have seven points in the plane. Prove that we can always select three

which do not form an isosceles triangle. For six points this does not necessarily hold. (If A, B, C are on a line we can define that they do not form an isosceles triangle if $AB \neq BC$.)

Solution by Michael Golomb, Purdue University. If every triangle formed by three of the n points, 1, 2, $\cdots$, n, is isosceles we call the figure an isosceles n-point.

Isosceles 4-points. 1. Let there be three equal sides issuing from one vertex, say $12=13=14$. Then 234 is an isosceles triangle and 1 is its circumcenter. Conversely, the vertices and circumcenter of any isosceles triangle form an isosceles 4-point (type I.)

2. Suppose there is no vertex from which issue three equal sides, and suppose $12=13$. We distinguish two cases: (a) $42=43$. Then since $14 \neq 12$ and $41 \neq 42$, we have $21=24$. Similarly, $31=34$. Hence the 4-point is a rhombus (type II). (b) $24=23$. Then $42=41$ and $34=31$. Hence $21=13=34=a$ and $14=42=23=b$. We have a trapezoid with three equal sides a and with base $b=\frac{1}{2}a(1+\sqrt{5})$. Hence 1, 2, 3, 4 are four vertices of a regular pentagon (type III).

Isosceles 5-points. Since four vertices of an isosceles 5-point must form an isosceles 4-point, we may show by similar elementary argument that every isosceles 5-point is of one of the three types: (I) a square with its center, or (II) the vertices of a regular pentagon, or (III) the center and four of the vertices of a regular pentagon.

Isosceles 6-points. Among the six points there must be either a square or four vertices of a regular pentagon. In the first case there is only one position left for the remaining two points, hence no isosceles 6-point is obtained. In the second case, there are two positions left for the remaining two points and we obtain the only isosceles 6-point: the vertices and center of a regular pentagon.

Isoceles 7-points. Among the seven points there must be the vertices of a regular pentagon. Since there is only one position left for the remaining two points, the isosceles 7-point is impossible.

Editorial Note. See also the solution and notes for No. E 735 (1947, 227–229).

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Concepts of Mathematics. By B. W. Jones. New York, The Macmillan Company, 1947. 13+294 pages. $4.00.

The author of this book faced a problem much more difficult than that of writing a text for a standardized course in introductory college mathematics. The book is one more attempt to present a terminal course for students of minimum training who want firmer grounding in such mathematics as may be useful or interesting to them in later life. In designing such a course not only must there be kept in mind the limited ability and even more limited training of the students concerned, but it must be realized that the present day applications of mathematics are so numerous and involve so many techniques that even a minimum list of topics is larger than can be effectively dealt with.

In his book Mr. Jones has treated at about equal length topics useful to his readers and those primarily of recreational interest to them. His choice of topics, as would that of any other writer, invites argument, for if mathematics is to be a part of a liberal education we must discover what domains of mathematics throw most light upon the nature of the discipline and what ones are of most use to persons of scant training. Mr. Jones has touched upon many areas of mathematics, including several of relatively recent discovery, such as groups, non-euclidean geometry and topology, and, as indicated below, has presented the material understandably, coherently and interestingly. But there are two central ideas in mathematics of such far-reaching importance that one wonders if their omission can be justified. The idea of an infinite sequence appears more or less incidentally in the paragraphs on the integers, on the decimal representation and on prime numbers. But infinity, as a concept, and the exciting and stimulating field of the infinite are passed by. Limits enter only with regard to the convergence of a geometric series; the vast domain of the calculus, with the limit at its heart and itself the heart of the most widely applied mathematics, is wholly neglected. The reviewer seriously questions whether anyone can claim even a casual acquaintance with mathematics unless he has at least met the basic ideas of the calculus.

The first chapter of the book is a brief discussion of logic, in which Boolean diagrams are introduced, the syllogism is treated (without naming it), and types of bad argument are exposed. Much more is covered than the few pages would indicate. The exercises, of which there are many, are varied and interesting, and should stimulate the thinking of any college freshman. In Chapter 2 the author first takes up a discussion of counting and one-to-one correspondences, defining the natural numbers as those we obtain by counting things, in particular, arrays of dots. The sum of two natural numbers is defined as the natural number of the

obvious compound array of dots. Using this definition the commutative and associative laws are established. Multiplication is then similarly treated. Subtraction and division are clearly defined in terms of addition and multiplication. It is emphasized that the difference and quotient are not always defined in the system of natural numbers. The remainder of the chapter is devoted to such topics as number system bases, the game of Nim, congruences, groups, prime numbers and tests for divisibility. The pace quickens in Chapter 3 and in rapid succession the author introduces the negative integers, the rationals, the irrationals and complex numbers. At each step the point is well made that the numbers introduced are new creations, invented to serve needs not served by those of previous categories. Although no attempt is made to establish in any completeness the fact that these new numbers satisfy the usual laws, it is emphasized that this fact requires proof. There is a good discussion of the decimal notation.

Chapter 4, entitled *Algebra*, covers such topics as square numbers, Pascal's triangle, compound interest, annuities, the binomial theorem, and linear and diophantine equations. The distinction between identities and conditional equations is emphasized, and the importance of the equivalence of equations well explained. The chapter abounds in exercises of both practical and recreational interest, including several of the puzzle type. Chapter 5 is concerned with graphical matters. Measurement is introduced as an ordering process, refinement being brought about through suitable choices of units of measurement. Data graphs are defined, the straight line and its equation are discussed and a few non-linear graphs are illustrated. Curve fitting by straight lines and parabolas is treated in connection with first and second differences. Frequency tables, the histogram, the mode, the median, and the three common means are defined. The treatment here is almost too brief to be of much value. In particular, although four pages are devoted to the five averages, there is little indication of the varying importance of each of these for different types of data. Permutations, combinations and simple combinatorial probability are treated in Chapter 6. Mathematical induction makes its sole entry in the derivation of the formula for ${}_nC_m$. There is reference to mathematical expectation and its application in gaming and insurance.

The remainder of the book is devoted to geometry and topology. In Chapter 7, entitled *Mirror Geometry*, reflection, as a particular type of geometric transformation, is the central idea. Rotation and translation are defined, and the three transformations are tied together by a statement and discussion of the theorem that the product of an even number of reflections is equal to that of a translation and a rotation. Definitions of euclidean congruence and the group of euclidean displacements close the chapter. Lorentz geometry, in which the admissible transformations are $x'=kx+c$, $y'=y/k+d$, forms the subject of Chapter 8. The invariance of incidence relations and the non-invariance of (euclidean) distance are shown. Congruence is defined and the congruence of non-equal angles is illustrated. Although very brief, the material is so presented

that any intelligent student should catch from it some glimpse of the richness of non-euclidean geometries. The final chapter, *Topology*, is one of the best from the standpoint of exposition. The linear graph is introduced via the Königsberg problem and the important simple theorems established. The four-color problem is defined and the six-color and five-color theorems proved. The chapter closes with brief paragraphs on planar networks, surfaces, the Möbius strip and maps on the torus.

There is a bibliography of nearly forty items to which the student is repeatedly referred for material for additional study. The bibliography is notable in being well annotated, a feature of too infrequent occurrence.

It is scarcely a criticism of the book to say that its classroom use requires superior teaching. But it must be emphasized that it is not a book which can be handed to the pedestrian teacher with the expectation that he will impart even the substance of the material let alone the spirit. The book is not adequate as a text in the hands of such a person; too much must remain unwritten which only an enthusiastic teacher and well-trained mathematician can supply in classroom discussion.

The reviewer's major criticism of the book is that it is flippant. One feels much too strongly that the author is merely trying to show that mathematics is fun. Mathematicians know their subject is fun, but they know also that it is a serious discipline; in fact, as any first-rate mathematician will witness, the real fun comes only when one recognizes the importance of mathematics and does serious thinking about it. Motivation is not wholly lacking in this book, but that for the really important ideas is scant and subordinated. One gathers that the author feels apologetic about the importance of his subject or that he is impatient to get on with the fun.

This flippancy is aided and abetted by one feature which distinctly mars the book. The author indulges in "asides" on almost every other page. These are intended to be humorous, and in the proper atmosphere they might be, but that atmosphere is the classroom and not the text. It is annoying when reading an otherwise cogent argument suddenly to have one's train of thought upset by a facetious reference to elements which form no proper part of the argument. The use of good English is not always observed; definitely awkward constructions are too frequent.

Although it is mentioned in the preface, the title page makes no reference to the fact that the book is a rewriting of an earlier set of notes which were the joint product of several members of the staff at Cornell University.

G. B. VAN SCHAACK

Intermediate Algebra. By R. S. Underwood, T. R. Nelson, and S. Selby. New York, The Macmillan Company, 1947. 7+283 pages. $2.60.

This is another in the growing list of textbooks written especially for college students in algebra who, for any reason, are not ready for the kind of course ordinarily given under the name of "College Algebra." It is unquestionably one

of the better books of the type and deserves consideration by anyone who is selecting a book for a course in intermediate algebra.

The discussions in the book have been written with a clarity and definiteness that should be very helpful to the students. They show a regard for mathematical accuracy that is not always found in books of this level. The number of problems should be ample for all but the very slowest classes, and the sets of problems are well graded in difficulty.

The book starts at the beginning of algebra and includes all the topics one would expect to find up through the simpler types of simultaneous quadratic equations, the factor theorem, the binomial theorem, variation, progressions and logarithms.

The publisher deserves credit for producing a book in which the typography is pleasing and the binding is quite attractive and apparently substantial.

W. C. McDaniel

A Handbook on Curves and Their Properties. By R. C. Yates. Ann Arbor, Edwards Brothers, Inc., 1947. 10+245 pages. $3.25.

This is a second edition of *Curves* by the same author which was reviewed in this Monthly, Vol. 54 (1946) pages 175–176. The chief changes from the first edition include a more substantial binding, an expanded index, additional and more complete references, and minor revisions of the text in the interest of greater accuracy and clarity.

C. B. Allendoerfer

Analytic Mechanics. (Prepared for the Department of Mathematics of the United States Naval Academy.) By A. E. Currier. Annapolis, The United States Naval Institute, 1948. 10+306 pages. $4.75.

This text was prepared to meet the requirements of the curriculum at the United States Naval Academy. The purpose, as stated in the preface, is to provide the student with an adequate understanding of mechanics to facilitate further study of the engineering and professional subjects prescribed at the Academy. In accomplishing these objectives the author has presented a well organized and concise treatment of analytic mechanics in two dimensions. A knowledge of elementary calculus and an introductory course in differential equations is assumed as a prerequisite for the reader. Indeed, the author should be commended for persistently emphasizing the role of mathematics in the derivation of the laws of Newtonian mechanics from the fundamental postulates (Newton's Laws). The definitions and the laws (theorems) are, in general, clearly and precisely stated. It is evident that a careful attempt is made to emphasize the assumptions upon which these laws as formulated are based and to enumerate the consequent limitations of the theory.

The order of presentation of the material is as follows: Chap. I. Introduction; Chap. II. Components, Sums, Moments and Equilibrium of Forces in the Plane; Chap. III. Physical Significance of a System of Forces in Equilibrium, and Ap-

plications to Statics Problems; Chap. IV. Trusses; Chap. V. Equivalent Forces, Equivalent Systems of Forces and Resultants; Chap. VI. Resultant of Gravity Forces, Center of Gravity, Center of Mass, Centroids; Chap. VII. Moment of Inertia; Chap. VIII. Normal Force and Force of Friction; Chap. IX. Strength of Materials, and Beams; Chap. X. Plane Dynamics of a Particle; Chap. XI. Work, Energy, and Potential Functions; Chap. XII. Motion of a System of Particles; Chap. XIII. Impulse and Momentum, Impulsive Forces.

The first nine chapters are devoted primarily to the theory of statics for two dimensional force systems, while the remaining four chapters treat the dynamics of particle systems and rigid bodies with plane motion. The chapter headings are in most cases sufficient to describe the contents of the chapter without the addition of further details. An excellent choice of topics is displayed throughout the text. In this respect, Chapter X on the dynamics of a particle is representative. It includes the following list of topics: 1. Assumptions Implied by Newton's Law of Motion; 2. Position, Velocity, and Acceleration; 3. The Hodograph of the Motion of a Point; 4. Motion of a Particle Acted on by the Force of Gravity, Assumed Constant; 5. Harmonic Motion with Damping; 6. Forced Harmonic Motion; 7. Tangential and Normal Components of Acceleration; 8. The Simple Pendulum; 9. Motion of a Particle Acted on by a Central Force Obeying the Inverse Square Law; 10. Effect of the Earth's Rotation, Coriolis and Centripetal Acceleration; 11. Ballistics: The Path of a Projectile.

One of the noteworthy features of the book as a text is the profusion of examples illustrating the theory. The clear explanation of the solutions of these examples and the detailed interpretation of the results will be very helpful to the student in understanding the theory and its application to the solution of problems. The student is given ample opportunity to become proficient in the solution of problems by the inclusion of an abundant set of well selected exercises. The answers are given to alternate problems. The excellent diagrams and figures illustrating the theory and the exercises are also worthy of special mention.

The format of the book is excellent. The proof has been carefully read, for there are remarkably few typographical errors.

R. C. F. BARTELS

The Early Work of Willard Gibbs in Applied Mechanics. By L. P. Wheeler, E. O. Waters, and S. W. Dudley. New York, Henry Schuman, Inc., 1947. 7+78 pages. $3.00.

The year 1947 was the first centennial anniversary of the founding of the Sheffield Scientific School of Yale University, and this little volume was published as a part of the commemoration of that occasion.

Josiah Willard Gibbs was one of the most illustrious graduates of Yale and was internationally known for his work as a mathematical physicist. The present volume, however, presents some early practical work that probably many people were unaware of. He graduated in 1863 with the degree of Doctor of Philosophy, his being the fifth such degree to be granted in the United States. The

subject of his dissertation was "On the Form of the Teeth of Wheels in Spur Gearing," but it was not printed during his lifetime.

That this was indeed his doctoral thesis is, however, a matter of inference and not of record. It was the contents of a manuscript, lacking title page and binding, which was found among the papers in the study at his home following his death. In this connection Professor Waters points out "the records of that time are confused and meager; there was no requirement that theses be published, and no complete file was kept in the Yale Library or elsewhere of theses that had been accepted." But the manuscript was examined by two of Gibbs' colleagues and they gave it as their considered opinion that it was his doctoral thesis.

The original manuscript, in Gibbs' own handwriting, was bound in 1925 and placed in the Rare Book Room of Yale University Library. It appears in printed form for the first time in this volume.

There is an Introduction contributed by Dr. Lynde Phelps Wheeler and a Commentary by Professor Everett Oyler Waters, both associated with Yale University.

Included in the book are also accounts of two other contributions of Gibbs in his early days in the practical realm. One is a discussion of An Improved Railway Car Brake by Dean Samuel Wilbur Dudley of Yale School of Engineering, and the other an account of The Gibbs Governor for Steam Engines by Dr. Wheeler.

The book throws an interesting sidelight on the early life and times of one of America's greatest scientists.

J. W. CAMPBELL

Les Principes Mathématiques de la Mécanique Classique. By M. Brelot. Grenoble, B. Arthaud, 1945. 62 pages. 120 fr.

This monograph is essentially a sketch of a modernized treatment of classical mechanics. The author considers that the usual treatments are wanting in clarity and precision, and he wishes to develop the subject in a new and concise way, on the basis of a clearly stated system of axioms. He indicates that the exposition that he has in mind would be somewhat comparable with the modern treatments of the theory of probability.

His first criticism of the traditional expositions relates to the procedure that is employed in advancing from the theory of systems consisting of finite sets of particles to the theory of systems consisting of continuous distributions of matter. He describes the practice of replacing the finite summations in the first theory by the integrations in the second theory as being "sans justification sérieuse." In order to avoid this difficulty he introduces the concept of additive set functions at an early stage, and states the dynamical formulae and theorems in terms of this concept. In brief, he postulates a point set, depending upon a parameter t, in Euclidean space, with various scalar and vector additive set functions defined on the set. These functions represent distributions of mass,

velocity, acceleration, force, and so forth. Then it turns out that it is possible to write the fundamental dynamical relations in general forms, which are immediately applicable to all particular cases.

Another of the author's criticisms relates to the way in which the notion of internal forces is usually treated. He emphasizes the fact that this notion is largely conventional, and he subjects the significance of the notion, and its applications, to a careful analysis.

At a point about half way through the booklet we find statements of the classical theorems concerning the existence and continuity of solutions of systems of differential equations. Most of the latter part of the work is devoted to a study of certain dynamical consequences of these theorems. Specifically, much of this part relates to the delicate problems presented by the motion of a system in the neighborhood of an equilibrium configuration.

The last few pages are devoted to a rapid discussion of percussions.

The author's general objective is one that will appeal strongly to all mathematicians and to many physicists. Furthermore, there will probably be general agreement that the specific proposals made are entirely sound. However, this sketch seems to be too summary to permit the implications and merits of the new ideas to become fully apparent. It is to be hoped that before long the author will expound his point of view in a full-length treatise.

L. A. MacColl

Curso de Matematica, en Forma de Problemas. By J. Gallego-Diaz. Madrid, Editorial Dossat, S. A., 1944. 12+344 pages. 60 pesetas.

This is a rather good collection of problems with solutions, except in a very few cases, in the subjects which are usually covered by undergraduate courses in American colleges. Some of them are in the nature of exercises, but most of them call for more than mere application of a few theorems; many of them are taken from examination papers set in some of the Spanish engineering schools in recent years. They are on the general level of the problems in the sections of this MONTHLY devoted to problems, elementary and advanced. In several instances important standard topics enter into the solutions; references to the literature would have been in order in such cases. The different topics are given treatments which vary a great deal in length. The longest section is that on limits, sequences, and infinite series and products, containing 44 problems. There are 25 problems in the section on conics, 18 on kinematics and differential geometry, 15 on probability, 10 miscellaneous problems, and groups of less than 10 each in sections on maxima and minima, transformations, quadric surfaces, and so on, several topics being represented by but a single problem.

The three problems which follow are quoted so as to give some idea of the general character of the contents of the book:

1. An urn contains one white ball and one black one. One ball is drawn and returned to the urn, with another ball of the same color as the one drawn. Con-

tinuing in this manner, after each drawing the number of balls will be increased by one. What is the probability that, when the urn contains 20 balls, there will be 10 white ones and 10 black balls? The problem is solved and generalized in obvious manner, with a reference to Polya's paper in vol. 1 of the *Annales de l'Institut Henri Poincaré.*

2. On a half line points, marked 0, 1, 2, $\cdots$, are laid off, equal distances apart. These points are projected on another line giving rise to the points $P_0, P_1, P_2, \cdots$; a uniform scale is established on this new line and in this scale the distances P_2P_5 and $P_{15}P_{25}$ are given. It is required to determine the limit of the length of the segment P_0P_n as n tends to infinity, and also the lengths in this scale, of the segments P_0P_k when P_k falls on a division point of the scale.

3. It is required to determine the conditions on 4 lines in space, R_1, R_2, R_3, and R_4, in order that, starting with an arbitrary point P_0 in space and constructing P_i as the mirror image of P_{i-1} in R_i, $i=1, 2, 3, 4$, the point P_4 will coincide with P_0.

This book could be very useful for college teachers of mathematics.

ARNOLD DRESDEN

Nomography. By A. S. Levens. New York, John Wiley and Sons, Inc., 1948. 8+176 pages. $3.00.

A nomograph, or nomogram, or alignment chart, is a graphical device for solving an equation in three unknowns for one of the unknowns when the values of the other two are given. One draws a straight line across the three scales which constitute the nomograph, and the three scale readings at the points of intersection yield a set of values for the unknowns which satisfies the equation. Not all equations lend themselves to this graphical device. An equation in four or more unknowns may sometimes be solved by the successive use of two or more nomographs.

The present book is intensely practical. The author chooses nine type equations of general interest and gives detailed instructions for making a convenient nomograph for each type. Although the emphasis is always on the practical construction, the author does give a geometric proof of the correctness of the nomograph for each case. The explanations are clear, and are accompanied by numerous examples and figures. The use of three-rowed determinants, which plays such a large part in the theory of nomographs, is briefly presented in ten pages. There is an appendix containing about thirty nomographs intended for practical use. There is also a bibliography of eighteen references. The book is well indexed.

This book may be recommended to one who wishes to make nomographs or teach the art of making them, provided his interest is altogether on the practical side of the subject. The author's nine types do cover most of the usual cases that arise in practice, and very little mathematical background is presupposed.

D. F. BARROW

The Strange Story of the Quantum; An Account for the General Reader of the Growth of the Ideas Underlying our Present Atomic Knowledge. By Banesh Hoffmann. New York, Harper and Brothers, 1947. 11+239 pages. $3.00.

Professor Hoffmann's subtitle gives a concise description of his aim, and of his achievement; he gives an excellent account of the growths of the concepts underlying quantum mechanics and atomic and nuclear theory. Although the author is a mathematician, at present Professor of Mathematics at Queens College, New York, he tells his story, and many of its details, without the use of mathematical formalism. The contents of the theories, their relationship, "parentage," and cross-fertilization are clearly outlined, so far as it is possible without the use of the mathematical tools. The relations of theory and experiment are stressed at many points.

The book, or, following the author, the drama, starts with Hertz's discovery of electromagnetic waves, that triumph of the predictions of classical theories. Then the "Strange Story" begins to unfold; in Chapter 2 "The Quantum is Conceived" an account of Planck's work is given. The photoelectric effect is discussed in Chapter 3; this chapter and the next "Tweedledum and Tweedledee" dealing with interference, bring home, right at the start, the crucial wave-particle dualism. Chapters 4 and 5 give an exposition of the Bohr theory, the Rydberg principle, and the limitations of the Bohr model.

In "Act II" the author proceeds to follow "The Exploits of the Revolutionary Prince." De Broglie's work and the strong impetus of relativity to the development of his thoughts is mentioned in Chapter 8. In Chapter 9, headed "Laundry Lists are Discarded," the author speaks about the Heisenberg-Born Matrix Mechanics, and tries to bring out the mathematical as well as the physical problems involved. This is one of the chapters where this reader had misgivings about the reactions of the "general reader." The following two chapters "The Ascetism of Paul" and "Electrons are Smeared," on the Dirac symbolical method and Schroedinger's wave equation, are high points of the book. In Chapter 12 "Unification" the equivalence of the wave and matrix theories is discussed, and in Chapter 13 "The Strange Denouement" leads to the resolution of the wave-particle paradox, and the role therein of the Heisenberg indeterminacy principle. The last chapter of "Act II," the "New Landscape of Science," describes the general consequences of the new concepts.

In the "Epilogue" the developments of the thirties are traced. The discovery of the new particles, the positron, the neutrino, the neutron, and the meson, and the success of the quantum mechanical theory of coping with them are described.

The book is very good and absorbing reading; it is a serious attempt to show the relationship of mathematics and physics, and to trace the growths and replacement of concepts, without using the mathematical tools of quantum mechanics. It is unfortunate that in so aptly constructed a book the story is a bit overdramatized in places. Sometimes this will not help the reader much, and

may lead to misunderstandings, for instance the use of the author's "principle of perversity."

Another point is the lack of units, say on page 22, when speaking of h the author says "Its value is a mere .000 000 000 000 000 000 000 000 006 6." Also one would have liked a stronger emphasis on the permanence of classical theory, as far as macroscopic quantities are concerned. Notwithstanding these few objections, it is a fascinating book, almost in a class with Jeans and Eddington and the other great classics of Popular Science. It will be good collateral reading in courses on the history of science.

F. T. ADLER

NEW BOOKS RECEIVED

Intermediate Algebra for Colleges. By W. L. Hart. Boston, D. C. Heath and Co., 1948. 7+316 pages. $2.50.

College Algebra. By T. S. Peterson. New York and London, Harper and Brothers, 1947. 8+334 pages. $2.50.

Algebra for College Students. By J. R. Britton and L. C. Snively. New York, Rinehart and Co., 1947. 11+529 pages. $3.00.

Higher Algebra. (Sequel to Higher Algebra for Schools). By W. L. Ferrar. New York, Oxford University Press, 1948. 6+320 pages. $5.00.

Essentials of Analytic Geometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1947. 4+259 pages. $2.80.

Analytic Geometry. By P. K. Rees and E. D. Mouzon. New York, Dryden Press, 1948. 28+305 pages. $2.75.

Analytic Geometry. By P. R. Rider. New York, The Macmillan Co., 1947. 10+383 pages. $3.25.

Solid Analytic Geometry. By J. M. H. Olmsted. New York, Appleton-Century Co., 1947. 13+257 pages. $4.00.

Plane Geometry. By D. T. Sigley and W. T. Stratton. New York, Dryden Press, 1948. 12+242 pages. $2.25.

Unified Calculus. By E. S. Smith, M. Salkover, and H. K. Justice. New York, John Wiley and Sons; London, Chapman and Hall, 1947. 10+507 pages. $3.50.

Modern Mathematics. By S. A. Walling and J. C. Hill. New York, The Macmillan Co., 1948. 6+153 pages. $1.00.

A Text-book of Mathematical Analysis. The Uniform Calculus and its Applications. By R. L. Goodstein. New York, Oxford University Press, 1948. 12+475 pages. $9.00.

Modern Operational Calculus. By N. W. McLachlan. New York, The Macmillan Co., 1948. 14+218 pages. $5.00.

Applied Differential Equations. By F. E. Relton. London, Blackie and Sons, Ltd., 1948. 6+264 pages. 20s net.

Introduction to the Differential Equations of Physics. By L. Hopf. Translated by Walter Nef. New York, Dover Publications, 1948. 5+154 pages. $1.95.

Cours de Mecanique Rationnelle. By J. Chazy. Paris, Gauthier-Villars. Vol.

I, *Dynamique du Point Materiel*. 482 pages, 900 fr.; Vol. 2, *Dynamique des Systemes Materiels*. 6+511 pages, 1100 fr.

Set Functions. By Hans Hahn and Arthur Rosenthal. Albuquerque, University of New Mexico Press, 1948. 9+324 pages. $12.50.

A Treatise on Set Topology. Part I. By R. Vaidyanathaswamy. Madras, S. Mahadevan, 1947. 6+304 pages. Rs 16-4.

Mathematics for Use in Business. By C. E. Hilbron. Boston, Houghton-Mifflin C., 1948. 7+472 pages. $3.50.

Mathematics of Finance. By J. B. Linker and M. A. Hill. New York, Henry Holt and Co., 1948. 8+175+83 pages. $2.90.

The Mathematical Theory of Finance. Revised Edition. By K. P. Williams. New York, The Macmillan Co., 1947. 10+274 pages. $4.50.

Theory of Experimental Inference. By C. W. Churchman. New York, The Macmillan Co., 1948. 11+292 pages. $4.25.

Sampling Inspection. By The Statistical Research Group, Columbia University. New York, McGraw-Hill Book Co., 1948. 20+395 pages. $5.25.

Studies and Essays. Anniversary Volume Presented to R. Courant. New York, Interscience Publishers, 1948. 470 pages. $5.50.

A Collection of Papers in Memory of Sir William Rowan Hamilton. The Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

Newton Tercentenary Celebrations. By E. N. da C. Andrade, H. J. Keynes, and Others. New York, The Macmillan Co., 1948. 15+91 pages. $3.00.

Tables of Spherical Bessel Functions. Vol. 2. Prepared by the Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1947. 20+232 pages. $7.50.

The Differential Analyser. By J. Crank. New York, Longmans, Green, and Co., 1948. 8+137 pages. $2.50.

The Theory of Mathematical Machines. Revised Edition. By F. J. Murray. New York, King's Crown Press, 1948. 9+139 pages. $3.00.

Proceedings of a Symposium on Large-Scale Digital Calculating Machinery. Vol. XVI. Cambridge, Harvard University Press, 1948. 29+302 pages. $10.00.

Heat Conduction. By L. R. Ingersoll, O. J. Zobel, and A. C. Ingersoll. New York, McGraw-Hill Book Co., 1948. 12+278 pages. $4.00.

Nuclear Forces. Part I. By L. Rosenfeld. New York, Interscience Publishers, 1948. 20+181 pages. $5.00.

Mathematics for the Consumer. By R. Schorling, J. R. Clark, and F. G. Lankford, Jr. Yonkers, New York, World Book Co., 1947. 10+438 pages.

Mathematics for Radio Engineers. By L. Mautner. New York, Pitman Publishing Co., 1947. 8+327 pages. $5.00.

Two-Dimensional Fields in Electrical Engineering. By L. V. Bewley. New York, The Macmillan Co., 1948. 14+204 pages. $5.50.

Introductory Chemical Calculations. By S. J. Smith. New York, The Macmillan Co., 1947. 7+144 pages. $1.00.

Traffic Performance at Urban Street Intersections. Technical Report No. 1. New Haven, Yale Bureau of Highway Traffic, 1947. 15+152 pages.

Sobre las Orbitas Aparentes de las Estrellas Dobles Visuales. By Enrique Vidal. University of Santiago, 1947. 8+58 pages.

The Scientists Speak. By Warren Weaver. New York, Boni and Gaer, 1947. 13+369 pages. $3.75.

Bibliographie des Multigrades avec quelques Notices Biographiques. By A. Gloden and G. Palama. Luxembourg, 1948. 4+64 pages.

Secret. By W. W. Stout. Detroit, Chrysler Corp., 1947. 67 pages.

Number Readiness in Research. By A. Riess. New York, Scott, Foresman, and Co., 1947. 70 pages. $1.00.

Math is Fun. By J. Degrazia. New York, Gesham Press, 1948. 159 pages. $2.75.

A Centenary of Marxism. Edited by Samuel Bernstein and the Editors of Science and Society. New York, Science and Society, 1948. 196 pages. $2.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College held two parties, a buffet supper, and a boat ride to Bear Mountain in addition to the regular meetings at which the following papers were presented:

Philosophical problems relating to statistics and probability, by Jerzy Neyman of the University of California

Self-inductive theorems, by Professor James Singer of Brooklyn College

Transcendental numbers, by Professor J. H. Bushey

Mathematics and art, by Marian Boykan

Number scales, by Florence Burg

The number π, by Cecile Cohen

Mobius strips, by Marcia Geiger

Pythagorean numbers, by Dorothy Beck

An introduction to map projections, by Marianne Weisz.

The club was invited to join the *Physics Club* of Hunter College to hear I. F. Ritter of New York University speak on *Soap films and minimal surfaces.*

The officers for 1946–47 were: President, Wilhelmina Fluhr; Vice-President,

Florence Myres; Secretary, Ceceil Cohen; Treasurer, Ruth Friedman; Interclub Committee Representative, Marcia Geiger; Publicity Chairman, Florence Burg; Faculty Adviser, Carolyn Eisele.

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which the following papers were presented by members:

The construction of Pythagorean triangles, by Joseph R. Berry
Fundamental concepts of analysis, by J. R. Flynn
Fundamental number theory, by E. J. Freeh
Nomographic or alignment charts, by R. J. Schweller
Exterior ballistics, by R. R. Luthman
Curve fitting, by J. R. Westerheide
The irrational number system, by J. D. Griffin
Paradoxes of the infinite, by N. A. Engler
The application of Fourier's series to electrical circuits, by P. F. Swift.

Faculty talks presented during the year were:

The transcendence of e, by Dr. K. C. Schraut
A civil engineer looks at mathematics, by Prof. J. J. Chamberlain, acting head of the University Civil Engineering department
Questions after the lecture, by Prof. Merriman of the University of Cincinnati.

During the first semester the club took a special field trip to the University of Cincinnati Observatory where Dr. Paul Herget, Director of the observatory, spoke on *An application of finite differences to the computation of orbits*. The annual banquet, attended by the president of the University, the Dean of Science, several Mathematics Club Alumni including three past presidents and other distinguished guests was held at the Wishing Well Inn.

The Dean of Science Award, consisting of two volumes of *Differential and Integral Calculus*, by R. Courant, given to the student who delivers the most interesting paper during each semester, was presented to R. J. Schweller for the first semester and to J. D. Griffin for the second semester. The mathematics club alumni awards for excellence in advanced mathematics were presented to J. R. Berry, a senior, and to Charles Keller, a junior.

Several members attended and assisted in the annual symposium and dinner of the National Mathematics Honor Society of Secondary Schools, which was founded under the auspices of the Club, and aided in the installation of an additional chapter at Elder High school of Cincinnati.

The first edition of a proposed annual Mathematics Club *Bulletin*, containing reprints of all the student papers delivered before the Club and over ninety pages in length, was edited by the secretary. A new constitution, drafted by the officers, was accepted by the members.

Officers of the Club during the year were: President, John R. Westerheide; Vice-President, Edward J. Freeh; Secretary, J. Dennis Griffin; Publicity Secre-

tary, Thomas P. Hanlon; Treasurer, Peggy Ens; Faculty Advisor, Dr. K. C. Schraut.

Pi Mu Epsilon, Duke University

Two meetings of the *North Carolina Alpha* Chapter of *Pi Mu Epsilon* were held this year. At the fall meeting Dr. J. H. Roberts presented the topic *Paradoxes and mathematics*. At the spring meeting Dr. H. Hotelling presented the topic *Applications of mathematics*.

Initiation was held at both the fall and the spring meetings. In the fall nine new members were taken into the fraternity, and in the spring there were forty-nine new members taken in. The officers elected for 1947–48 are: Director, Jo Anne Walker; Vice-Director, Nancy Bloom; Treasurer, Wayne Bainbridge; Secretary, Jean Bellingrath.

Mathematics Club, Swarthmore College

The *Mathematics Club* of Swarthmore College, having lapsed into inactivity during the war, was revived last year and is now in its second year of existence. Meetings are held on alternate Thursdays, at which papers on topics of interest are given by students and faculty members. The object of these talks is to acquaint the students with areas of mathematics not taken up in the regular curriculum and to stimulate students to investigate these subjects further.

Papers given during the semester were:

Partition of numbers, by Dr. Rademacher of the University of Pennsylvania

Cryptanalysis, by Paul Mangelsdorf

Laplace transformations, by Dr. Elmore

Group theory, by Carl Levinson

Calculus of variations, by Irving Dayton

Finite fields, by Dr. Carruth.

The club also sponsors a problem contest every semester in the belief that active participation is essential to the study of mathematics. The winners of this term's contest are: Ned Freeman, Carl Levinson, Douwe Yntema, and Bob Kuller.

Officers for the semester were: President, George Yntema; Vice-President, Carl Levinson; Secretary, Patricia Plank; Treasurer, Nancy Burnholz.

For the spring semester the officers are: President, Carol Levinson; Vice-President, Edward Rawson; Secretary, Charlotte Garceau; Treasurer, Nancy Burnholz.

Mathematical Society, The University of Adelaide

The *Mathematical Society* of the University of Adelaide was conducted as a Colloquium or Seminar on the *Theory of the Gamma function* and some of its applications. The lectures of the students were prepared from typed notes on the subject by Prof. H. Schwerdtfeger. In 13 meetings the theory of the real Gamma function was covered as far as it is treated in *Cours d'analyse* by de la Vallée

Poussin, but using the method of E. Artin's *Einfuehrung* in the *Theorie der Gammafunktion* (Leipzig 1931). As an application we have dealt with Riemann-Liouville's Fractional Integration, using various sources; e.g., papers by J. D. Tamarkin (Annals of Mathematics 31, 1930, 219–229), W. L. Ferrar, (Proc. Royal Society Edinburgh 48, 1927–28, 92–105) and H. T. Davis (American Journal of Mathematics 46, 1924, 95–109).

The principle collaborators with Prof. Schwerdtfeger were: A. T. James, B.Sc., N. Munroe, B.Sc., R. B. Potts, B.Sc., Mrs. M. Sved, G. E. Wall, B. W. Worthley, B.Sc.

A colloquium on the *Theory of Matrix Functions* is contemplated along similar lines for the next school session.

Prof. Schwerdtfeger states that "the concentration upon a single subject defined in advance has, in fact, reduced the popularity of the mathematical society; but in return it gave more satisfaction to the more serious kind of students. It was therefore agreed to continue the society in this manner, changing the subject from year to year. This seems to be the more desirable under the circumstances."

Mathematics-Physics Club, College of Saint Teresa

The theme for the activities of the Mathematics-Physics Club of College of Saint Teresa for 1946–47 was *measurement*. Applications were made of the cross-staff, drum-head, clinometer, visor, range-finder, scout staff, square and cross staff, mirror, and yardstick.

Demonstrations of an oscillograph and a student-constructed radio as well as films on the parabola and crystallization were shown. Progressive mathematical games, a Christmas cryptograph, and quotations of mathematicians furnished digression at other meetings.

The year's activities were closed with a mathematical scout picnic at which the participants were tested in mapping, surveying, estimating measurements, the compass, the scout code, and laws. Appropriate badges were awarded.

The officers for 1946–47 were: President, Jean Stephany; Vice-President Gladys Schmitz; Secretary, Mary Vroman; Treasurer, Clarice Abts.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service, Princeton, New Jersey is offering for 1949–50 two research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. These fellowships carry a stipend of $2,200 a year.

They are available to men who are acceptable to the Princeton University Graduate School and who show exceptional aptitude for advanced training in psychology, psychological measurement and statistics.

Applications must be received in Princeton by January 15, 1949. Application blanks may be obtained from the Director of Psychometric Fellowship Program, Educational Testing Service, Box 592, Princeton, New Jersey.

SYMPOSIUM ON MODERN CALCULATING MACHINERY

A Symposium on Modern Calculating Machinery and Numerical Methods was held at the University of California at Los Angeles on July 29–31, 1948. It was sponsored by the Departments of Astronomy, Engineering, and Mathematics of the University and by the Institute for Numerical Analysis in cooperation with the Association and other scientific societies. The program included reports on computing machine developments in various research centers and on numerical methods in applied mathematics.

PERSONAL ITEMS

Professor W. D. Reeve, retiring head of the Department of Mathematics of Teachers College, Columbia University, has been honored by the establishment of the William David Reeve Scholarship in the Teaching of Mathematics.

Carnegie Institute of Technology anounces the following appointments for the year 1948–49: Professor F. D. Murnaghan as visiting professor for the first semester and Professor E. R. Lorch as visiting professor for the second semester; Dr. Abraham Charnes of the University of Illinois to an assistant professorship; Assistant Professors H. J. Greenberg and George Handelman of Brown University to assistant professorships.

Cornell University announces: Associate Professor C. J. Thorne of the University of Utah has been appointed Visiting Associate Professor; Associate Professor R. J. Walker has been promoted to a professorship; Assistant Professor Harry Pollard has been promoted to an associate professorship; Professor W. B. Carver has retired.

At Kent State University Assistant Professor R. Y. Iwanchuk of St. Basil's College and Mr. B. B. Dressler have been appointed to assistant professorships.

Rutgers University makes the following announcements: Mr. C. W. Saalfrank of Franklin and Marshall College, Mr. A. G. Makarov of the University of Pennsylvania, and Mr. D. A. Darling of University College have been appointed to assistant professorships; Mr. J. N. Livingood and Mr. E. D. Nering have resigned.

The University of Michigan announces the following promotions: Associate Professor S. B. Myers to a professorship, Assistant Professor R. M. Thrall to an associate professorship, Mr. P. S. Jones to an assistant professorship.

The University of Wyoming announces: Dr. W. N. Smith has been appointed Assistant Professor of Mathematics; Assistant Professor S. R. Smith has been promoted to an associate professorship.

Dr. H. G. Apostle, formerly of Amherst College, has been appointed to an associate professorship at Grinnell College.

Professor A. S. Besicovitch is serving as visiting professor at the University of Pennsylvania during the academic year 1948–49.

Dr. P. T. Bateman of Yale University is now at the Institute for Advanced Study.

Assistant Professor Truman Botts of the University of Delaware has been appointed to an acting assistant professorship at the University of Virginia.

Professor Louis Brand of the University of Cincinnati has received an appointment as visiting professor at the University of Hawaii for the year 1948–49.

Dr. C. T. Bumer of the Massachusetts Institute of Technology has been appointed Professor of Mathematics and Chairman of the Department of Mathematics at Clark University.

Associate Professor W. J. Combellack of Northeastern University has been appointed Professor of Mathematics and Head of the Department of Mathematics at Colby College.

Assistant Professor W. W. Dolan of the University of Oklahoma has accepted an appointment as professor and head of the Department of Mathematics at Linfield College.

Miss Geneva E. Durham of Atlantic Union College has been appointed Assistant Professor of Mathematics and Astronomy at Pacific Union College.

Assistant Professor Howard Eves of Oregon State College has been promoted to an associate professorship.

Professor H. P. Fawcett of Ohio State University has been appointed Chairman of the Department of Education.

Dr. R. D. Gordon of Indiana University has been appointed to an assistant professorship at the University of Buffalo.

Assistant Professor F. G. Graff of Amherst College has been appointed to an assistant professorship at Oberlin College.

Professor E. H. Hadlock of Hastings College has been appointed to an associate professorship at the University of Florida.

Dr. G. P. Hochschild of Harvard University has been named Assistant Professor of Mathematics at the University of Illinois.

Assistant Professor Carl Holtom of Purdue University has accepted an appointment as associate professor of mathematics at the U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio.

Dr. L. H. Kanter of the University of Wisconsin has been appointed to an assistant professorship at the University of Arkansas.

Dr. Samuel Kaplan has been appointed Assistant Professor of Mathematics at Wayne University.

Professor C. E. Melville has retired from his position as professor of mathematics and chairman of the Department of Mathematics after forty-six years of teaching at Clark University.

Professor G. W. Mullins of Columbia University has been given the title of Emeritus Professor of Mathematics.

Dr. Edward Paulson of the University of Washington has been promoted to an assistant professorship.

Miss Lucille K. Pinette of Colby College has been promoted to an assistant professorship.

Mr. I. H. Rose of Pennsylvania State College has been appointed to an assistant professorship at the University of Massachusetts.

Associate Professor Wladmir Seidel has obtained a year's leave of absence from the University of Rochester in order to work at the Institute for Numerical Analysis, University of California at Los Angeles.

Dr. Shien-Sin Shu has received an appointment as associate professor of mathematics and research associate in mechanics at Illinois Institute of Technology.

Professor R. R. Shumway of the University of Minnesota has retired.

Professor E. R. Sleight of Albion College has retired.

Assistant Professor W. D. Temple, Agricultural and Mechanical College of Texas, has been appointed to an associate professorship at Louisiana Polytechnic Institute.

Brother L. Thomas is now a member of the faculty of Christian Brothers College, Memphis, Tennessee.

Assistant Professor E. P. Vance has been appointed Chairman of the Department of Mathematics at Oberlin College for the year 1948–49.

Dr. W. E. Wilson has been appointed President of the South Dakota School of Mines and Technology.

The following appointments to instructorships are announced:

Dartmouth College: Mr. D. G. Dickson

Franklin and Marshall College: Mr. J. R. Holzinger

Illinois Institute of Technology: Mr. R. B. Brady

Kent State University: Mr. A. P. Boblett, Mr. W. C. Lowry, Mr. F. R. Olson, Mr. E. T. Stapleford, Miss Jane A. Uhrhan.

Rutgers University: Mr. W. W. Boone

University of Arkansas: Mr. W. C. Guenther

University of Buffalo: Miss Janet E. Abbey

University of Oregon: Mr. W. L. Shepherd

Lehigh University announces the appointment of Mr. R. R. Townsend to a graduate assistantship.

Professor Mae R. Andersen of Concordia College died on Marsh 28, 1948.

Sister M. Borgia Clarke of Webster College died on January 21, 1948. She had been a member of the Association for twenty-five years.

Professor Emeritus R. M. Ginnings of Western Illinois State Teachers College died on June 19, 1948. He was a charter member of the Association.

Assistant Professor T. R. Long of the University of Rochester died on July 11, 1948.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy persons have been elected to membership by the Board of Governors on applications duly certified:

Rev. H. B. Albiser, S.S.E., A.B. (St. Michael's) Instructor, St. Michael's College, Winooski, Vt.

P. M. Anselone, Student, College of Puget Sound, Tacoma, Wash.

L. G. Arnold, B.A. (Oberlin) Graduate Student, University of Michigan, Ann Arbor, Mich.

H. G. H. Bartram, B.A. (Colorado) Part-time Instructor, University of Colorado, Boulder, Colo.

V. N. Behrns, B.A. (Buffalo) Instructor, University of Buffalo, N. Y.

E. J. Berger, M.A. (Colorado State) Instructor, Monroe High School and College of St. Catherine, St. Paul, Minn.

Lipman Bers, Ph.D. (Syracuse) Asso. Professor, Syracuse University, N. Y.

Jacob Borsuk, B.A. (Brooklyn) Mathematician, U. S. Coast & Geodetic Survey, N. Y.

Leonard Bristow, Ph.D. (Illinois) Asst. Professor, University of Wyoming, Laramie, Wyo.

C. G. Burger, Jr., B.S. (Rensselaer Polytechnic Institute) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

Nikoline A. Bye, A.M. (Michigan) Asst. Professor, Central Michigan College of Education, Mount Pleasant, Mich.

Helen Selfridge (Mrs.) Clucas, M.A. (Oregon) Public Assistance Worker, Kern Country Welfare Dept., Bakersfield, Calif.

W. B. Conroy, Jr., M.A. (St. Lawrence) Asst. Professor, Clarkson College, Potsdam, N. Y.

Violet B. Davis, M.A. (Toledo) Asst. Professor, University of Toledo, Ohio

Rev. H. F. DeBaggis, C.I.C., Ph.D. (Notre Dame) Instructor, University of Notre Dame, Ind.

M. W. De Jonge, M.A. (Illinois) Asst. Professor, Purdue University, Lafayette, Ind.

B. K. Dickerson, M.A. (Northwestern) Instructor, Syracuse University, N. Y.

W. M. Duke, Sc.M. (New York) Chairman of Research Division, Cornell Aeronautical Laboratory, Buffalo, N. Y.

M. B. Evans, B.S.E. (Michigan) Instructor, Missouri School of Mines and Metallurgy, Rolla, Mo.

Ky Fan, D.Sc. (Paris) Asst. Professor, University of Notre Dame, Ind.

W. E. Felling, B.S.E.E. (St. Louis) Graduate Student, St. Louis University, Mo.

H. H. Fox, M.A. (Wisconsin) Assistant, University of Illinois, Urbana, Ill.

C. V. Fronabarger, M.A. (Geo. Peabody) Asst. Professor, Southwest Missouri State College, Springfield, Mo.

J. B. Garrett, Student, Siena College, Loudonville, N. Y.

Sidney Glusman, M.A. (Columbia) Instructor, Seton Hall College, South Orange, N. J.

N. H. Gottesman, B.S. (Notre Dame) Teaching Fellow, University of Notre Dame, Ind.

G. F. Guilford, Jr., M.S. (Syracuse) Asst. Professor, Rensselaer Polytechnic Institute, Troy, N. Y.

Smith Higgins, Jr., M.S. (Notre Dame) Instructor, University of Notre Dame, Ind.

H. K. Hilton, M.A. (Northwestern) Instructor, University of Wyoming, Laramie, Wyo.

K. E. Hoggatt, Student, College of Puget Sound, Tacoma, Wash.

V. E. Hoggatt, Jr. Student, College of Puget Sound, Tacoma, Wash.

R. J. Howerton, M.S. (Northwestern) Instructor, Regis College, Denver, Colo.

N. R. Hughes, A.B. (Wabash) Instructor, Wabash College, Crawfordsville, Ind.

T. R. Humphreys, M.A. (Oregon) Professor, Bergen Junior College, Teaneck, N. J.

R. E. Huston, Ph.D. (Chicago) Professor, Rensselaer Polytechnic Institute, Troy, N. Y.

T. C. HUTCHISON, Student, Pennsylvania State College, State College, Pa.

RUTH E. KENNEDY, B.S. (Louisiana Polytechnic Institute) Instructor, Louisiana Polytechnic Institute, Ruston, La.

P. J. KIERNAN, A.M. (Columbia) Instructor, Lawrenceville School, N. J.

F. P. KOWALEWSKI, JR., M.A. (Buffalo) Instructor, University of Buffalo, N. Y.

SIDNEY KRAVITZ, M.Ae.E. (New York) Mathematician, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

W. G. LEAVITT, Ph.D. (Wisconsin) Instructor, University of Nebraska, Lincoln, Neb.

J. V. LIMPERT, A.M. (Syracuse) Instructor, Syracuse University, N. Y.

R. L. LOKENSGARD, Ed.D. (Columbia) Head of Dept., Winona State Teachers College, Minn.

R. B. MCHUGH, B.A. (Minnesota) Teaching Asst., University of Minnesota, Minneapolis, Minn.

R. A. MILLER, M.A. (Mississippi) Asst. Professor, University of Mississippi, University, Miss.

TEXAS MORRIS (Miss) 1516 Chestnut St., Wilmington, N. C.

HELEN DOROTHY MORTELL, M.A. (Catholic University) Instructor, Catholic University of America, Washington, D. C.

F. R. OLSON, B.A. (Alfred) Instructor, Hamilton College, Clinton, N. Y.

S. R. ORR, B.A. (Hiram) 129 North Elm St., Columbiana, Ohio

FLORENCE E. POOL, M.A. (Nebraska) Instructor, University of Nebraska, Lincoln, Neb.

W. G. PREBLE, B.S. (Tulane) Computer, Corps. of Engineers, U. S. Army, New Orleans, La.

HENRY RAINBOW, B.A. (Cambridge) Shell Research Laboratory, Bellaire Road, Houston, Tex.

R. R. REED, A.B. (Union) Instructor, Clarkson College, Malone, N. Y.

G. X. SALTARELLI, M.S. (Notre Dame) Teaching Fellow, Notre Dame University, Ind.

A. C. SCHAEFFER, Ph.D. (Massachusetts Institute of Technology) Professor, Purdue University, Lafayette, Ind.

SISTER MARY VIRGILIA DRAGOWSKI, O.S.F., Ph.B. (Detroit) Student, University of Notre Dame, Ind.

SISTER MARY FERRER MCFARLAND, M.S. (De Paul) Student, University of Notre Dame, Ind.

RUBIN SMULIN, B.S. (Miami) Electrical Engineer, Unity Electric Co., Inc., Miami Beach, Fla.

F. M. STEIN, M.S. (State University of Iowa) Instructor, Iowa Wesleyan College, Mount Pleasant, Iowa

EUGENE STEPHENS, M.S. (Washington) Asst. Professor Emeritus, Washington University, St. Louis, Mo.

R. A. STRUBLE, B.S. (Notre Dame) Graduate Assistant, University of Notre Dame, Ind.

ANNA K. SUTER, A.M. (Indiana) Instructor, Purdue University Extension Division, Indianapolis, Ind.

A. K. TERZUOLI, M.S. (New York) Instructor, Polytechnic Institute of Brooklyn, N. Y.

W. J. THRON, Ph.D. (Rice) Asst. Professor, Washington University, St. Louis, Mo.

G. L. TILLER, Ph.D. (Kentucky) Asst. Professor, Utica College of Syracuse University, Utica, N. Y.

G. R. TROTT, Ph.D. (Johns Hopkins) Professor, University of Mississippi, University, Miss.

ELEANOR B. WALTERS, M.A. (Duke) Acting Head of Dept., Delta State Teachers College, Cleveland, Miss.

N. M. WATERMOLEN, B.S. (St. Norbert) Assistant, University of Wisconsin, Madison, Wis.

H. F. WILSON, M.S. (Oregon State) Director of Research, Pickett & Eckel, Inc., Alhambra, Calif.

W. J. YOUDEN, Ph.D. (Columbia) Mathematician (Statistical), National Bureau of Standards, Washington, D. C.

MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at The Citadel, Charleston, South Carolina, on Friday and Saturday, March 19–20, 1948. Professor J. W. Cell, Chairman of the Section, presided at the Friday afternoon and Saturday morning meetings, except for the meetings of the subsections, which were presided over by Major

L. A. Dye, Vice-Chairman, and Professor H. K. Fulmer. Colonel C. F. Myers, Jr., of The Citadel presided on Friday evening at the informal dinner given in honor of the visiting speaker, Lt. Colonel R. C. Yates.

After the dinner, an informal discussion was led by Professor Tomlinson Fort on *Common Problems Due to Overcrowding, Poor Preparation, and Inexperienced Instructors.*

There were about one hundred and fifty present, including the following seventy-eight members of the Association: R. H. Ackerson, Louise Adams, G. E. Albert, J. C. Barnes, D. F. Barrow, Helen Barton, W. S. Beckwith, R. C. Blackwell, R. G. Blake, J. P. Brewster, N. R. Bryan, Berdie J. Buffkin, R. C. Bullock, E. A. Cameron, J. W. Cell, B. G. Clark, A. C. Cohen, J. B. Coleman, W. J. Conner, R. W. Cowan, Nelle C. Douglas, Jeanette R. Durst, L. A. Dye, R. B. Folsom, Tomlinson Fort, Jack Frierson, H. K. Fulmer, L. L. Garner, W. W. Graham, C. L. Hair, E. A. Hedberg, R. A. Hefner, C. W. Hook, H. B. Hoyle, Jr., V. A. Hoyle, G. B. Huff, L. P. Hutchinson, J. A. Hyden, Rosa L. Jackson, F. W. Kokomoor, G. B. Lang, J. W. Lasley, Jr., R. J. Levit, Anne L. Lewis, F. A. Lewis, Nathaniel Macon, C. F. Martin, W. J. Mays, S. W. McInnis, W. G. Miller, J. D. Novak, W. V. Parker, William Pennington, Jr., R. I. Pepper, Lillian Perkins, Mary Pettus, C. G. Phipps, Alice B. Rabon, Ellen F. Rasor, B. P. Reinsch, G. E. Reves, H. A. Robinson, L. V. Robinson, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, C. Eucebia Shuler, T. M. Simpson, W. B. Stovall, Jr., Cora Strong, C. S. Sutton, J. M. Thomas, R. Z. Vause, J. A. Ward, W. W. Weber, W. L. Williams, R. C. Yates, G. C. Zader.

At the business session the following officers were elected for the coming year: Chairman, L. A. Dye, The Citadel; Vice-Chairman, F. A. Lewis, University of Alabama; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its March 1949 meeting at the University of Alabama.

The program consisted of the following papers:

1. *Dissection of the euclidean plane by the conic sections*, by Lt. W. J. Conner, The Citadel.

The plane is divided into not more than $2n^2-n+1$ regions by n parabolas. Similar formulas were obtained for the other conic sections.

2. *On the characteristic equations of certain matrices*, by Professor W. V. Parker, University of Georgia.

This speaker generalized a theorem proved by Brauer, and published under the same title in the *Bulletin of the American Mathematical Society*, Vol. 52, pp. 605–607. The generalized theorem was stated as follows: Let A, C_1, and C_2 be n-rowed square matrices such that $C_1A=AC_2=0$. If $C=C_1+C_2$, and if B is an arbitrary n-rowed square matrix, then AB and $A(B+C)$ have the same characteristic equation.

3. *The effect of hyperbolic grooves on the stress distribution in a wood plate*, by Professor C. B. Smith, University of Florida, introduced by H. A. Robinson.

This paper consisted of an analysis of the stress distribution existing in a wood plate of large length with hyperbolic grooves. Assuming a wood plate had two perpendicular axes of elastic symmetry in the plane of the plate, the analysis furnished a formula for the factor of stress concentra-

tion, that is, the ratio of the maximum stress occurring at the bottom of the grooves to the average tensile stress over the cross section.

4. *The exponential function and the Laplace transform*, by Professor J. W. Cell, North Carolina State College.

With this expository paper, some fifty slides were shown which illustrated the following ideas: (1) the meaning of the transformation $x'=bx$ and $y'=y/a$ to reduce $y=ae^{bx}$ to the simpler form $y=e^x$ (both graphical meaning and the interpretation in terms of dimensionless variables); (2) applications of this law of growth or decay to many fields of science and technology; (3) functions related to the exponential function by compounding this function with other basic functions; (4) the meaning of the Laplace transformation, with graphical illustrations showing the transforms of several elementary functions; (5) types of problems to which the Laplace transform method applies.

5. *Groups of birational transformations transitive on the rational points of a plane curve*, by Professor G. B. Huff, University of Georgia.

It was shown that for any elliptic cubic curve, there exists a group $G(c)$ of Cremona transformations which is transitive on the rational points of the cubic, and which has a finite number of generators. These results were capable of being generalized in several directions.

6. *The expansion of x^n in terms of $\binom{x}{j}$*, by Professor G. B. Lang, University of Florida.

Explicit formulas were obtained for the expansion. The results included a solution to Problem 4276 in this MONTHLY, vol. 54, 1947, p. 601.

7. *A generalization of Taylor's theorem*, by Professors C. L. Seebeck, Jr., and P. M. Hummel, University of Alabama.

A set of series expansions for functions of one variable which reduced to Taylor's series with remainder as a special case was developed. For a symmetric case, the error for n terms of the series was found to be less than that for a Taylor's series of $2n$ terms.

8. *Quasi-monotonic series*, by Professor Tomlinson Fort, University of Georgia.

Series of the type $a_1+a_2+a_3+\cdots$, where $a_n \geqq a_{n+1} \geqq 0$ are known as monotonic series. Such series have been extensively studied and many properties developed. The present paper considered certain generalizations and analogues of the relationship $a_n=a_{n+1}>0$ which assures for the series many of the well known properties of monotonic series.

9. *Some properties of plane curves*, by Lt. Colonel R. C. Yates, United States Military Academy.

Colonel Yates discussed the following topics: (1) folding and creasing the conics; (2) the projection upon a focal radius of the normal length N of a conic (constant and equal to the semi-latus rectum); (3) instantaneous centers and the construction of tangents to such curves as the ellipse, limaçon, conchoid, strophoid, and cycloid; (4) caustics, particularly the evolute of a central conic formed by light rays refracted through a plane surface; (5) the rose curves and their generation as special hypotrochoids; (6) the carpenter's square and its use in describing the conics, cardioid, strophoid, cissoid, and conchoid.

10. *Determinants and undergraduate mathematics*, by Major L. A. Dye, The Citadel.

The content and methodology of most undergraduate courses in mathematics are long overdue for a critical evaluation as to their justification and merit. It was suggested that the study and use of determinants in courses up to and including the calculus was an essential waste of time.

11. *An expression for a general law of mortality*, by Mr. W. J. Mays, Assistant Actuary, Liberty Life Insurance Company.

It was assumed that the probability of joint survival of m lives over any period of time may be expressed as a function of n variables ($n<m$). This assumption led to a set of linear differential equations involving the probabilities of survival of single lives. The solution was remarkable in that it comprehended almost every function that had been used successfully on empirical grounds to represent the mortality function l_x, that is, the number of persons who attain exact age x according to the mortality table.

12. *Imaginary graphs of elementary real functions of "excluded" portions of the x-axis*, by Professor J. W. Ward, University of Georgia.

Let the X, Y_r and Y_i axes be three mutually perpendicular axes in space. Real values of y are plotted along the Y_r-axis, and imaginary values of y along the Y_i-axis. The coördinate X is restricted to be real. The real graph of the real function $f(x, y)=0$ will lie in the XY_r-plane. Real values of x that give pure imaginary values of y will give points in the XY_i-plane. For example, $x^2-y^2=1$ is a hyperbola in the XY_r-plane, and a circle in the XY_i-plane. Professor Ward showed the imaginary parts of several elementary functions that are generated by "excluded" values of x.

13. *On the Lagrange multiplier*, by Professor C. G. Phipps, University of Florida.

This speaker considered the Lagrange multiplier as a constant, as a function of the variables involved, and as a function of certain parameters. The usual manipulation was justified; the multiplier was interpreted geometrically; and finally the problem of working backwards from the first results was discussed.

14. *On the sign of certain minors arising in the expansion of a Vandermonde determinant*, by Dr. R. J. Levit, University of Georgia.

In the theory of extremal properties of functions on a finite set it is important to determine the algebraic sign of certain determinants closely related to Vandermonde's. In this paper it was shown by an indirect method that the determinants are all non-negative.

15. *Iterations of quadratic polynomials*, by Lt. C. S. Sutton, The Citadel.

A set D, associated with any quadratic polynomial $P(x)$ is defined as the set of all points in the complex plane whose iterates under $P(x)$ are uniformly bounded. The author proved that D is connected if and only if it contains the point $P(\rho)$, where ρ is the zero of the equation $P'(x)=0$, and that otherwise D consists of a non-denumerable number of closed disjoint sets.

16. *Two dimensional Riemannian spaces that admit continuous groups of homothetic transformations*, by Mr. E. B. Shanks, Vanderbilt University.

It was shown that there exists only a one parameter family of Riemannian spaces admitting a two parameter group, while euclidean space alone has a group associated with it with a greater number of parameters than two. The latter admitted a four parameter group.

17. *Some examples in which the Pearson criteria for classifying frequency curves break down*, by Professor A. C. Cohen, Jr., University of Georgia.

The Pearson criteria for typing frequency curves indicate that the distributions of sample

means from various Pearson populations are of the same type as the populations from which the samples are selected. In this paper, certain problems connected with the classification of frequency distributions were discussed. By employing the Pearson moment recursion formulas, it was demonstrated that the typing criteria break down in classifying distributions of sample means from Type II and Type VII populations.

18. *On properties of polynomial solutions of certain linear differential equation*, by Dr. R. W. Cowan, University of Florida.

The general solution of the differential equation was obtained by Frobenius' method. The infinite series thus obtained reduced to polynomials for certain integral values of the parameter. Dr. Cowan established an orthogenality relation, and the integrated square of the polynomials was evaluated.

19. *Some peculiar integral operators*, by Professor L. V. Robinson, University of South Carolina.

Two of the operators discussed were $x \pm D^{-1}$, and the third was $(x-D^{-1})(x+D^{-1})$. Interesting properties were found, especially for the third.

20. *The integration of* $\csc^n v\, dv$, by Professor W. G. Miller, Clemson College.

The author developed methods of integration involving odd powers of the secant and cosecant by the use of trigonometric manipulation only, the primary value of which was to establish a continuity in an elementary calculus course.

21. *Formulas for expressing a function of three variables in nomograph form*, by Professor D. F. Barrow, University of Georgia.

Professor Barrow gave two formulas for expressing a function of x, y, z in the form of a 3-rowed determinant, the elements of the first, second, and third rows being functions of x alone, of y alone, and of z alone, respectively (provided such an expression was possible). This was the first step in constructing a nomograph for the solution of the equation obtained by equating the function to zero.

H. A. ROBINSON, *Secretary*

SPRING MEETING OF THE MICHIGAN SECTION

The Spring meeting of the Michigan Section of the Mathematical Association of America was held in conjunction with the meeti ıg of the Michigan Academy of Science, Arts, and Letters at the University of Michigan at Ann Arbor on Saturday, April 3, 1948. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon sessions and a luncheon-business meeting were held, at all of which the Chairman, Professor Harold Blair, presided.

About ninety persons attended the meeting including the following fifty-four members of the Association: N. H. Anning, J. W. Baldwin, J. H. Bell, H. L. Black, Harold Blair, W. M. Borgman, J. W. Bradshaw, D. M. Brown, W. H. Cain, R. E. Carr, R. V. Churchill, C. J. Coe, A. H. Copeland, P. C. Cox, M. L. DeMoss, P. S. Dwyer, C. M. Erickson, F. D. Faulkner, C. H. Fischer, J. W. Foust, J. S. Frame, G. W. Grotts, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, E. E. Ingalls, L. A. Jehn, L. S. Johnston, Wilfred Kaplan, A. E. Lampen, H. D. Larsen, G. E. Markle, E. D. McCarthy, L. E. Mehlenbacher, D. C. Mor-

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The following officers were elected for the coming year: Chairman, Professor B. M. Stewart, Michigan State College; Secretary-Treasurer, L. J. Rouse, University of Michigan.

At the morning and afternoon sessions the following program of seven papers was presented:

1. *Polygenic functions and monogeneity*, by Professor Emeritus V. C. Poor, University of Michigan, introduced by the Secretary.

This paper includes a fundamental theorem on polygenic functions and a discussion of a certain type of monogeniety of such functions. A rather explicit characterization of such monogenic functions is exhibited in a particular theorem, and the dual relation between this and a second class of monogenic functions is considered.

2. *Some elementary proofs concerning binomial coefficients*, by Professor Fritz Herzog, Michigan State College.

Let p be a given prime and $c_{n,k}$ be a given binomial coefficient. The familiar formula for the highest power of p dividing $c_{n,k}$ can be interpreted in an elementary way in terms of the addition of k and $n-k$, written in the scale of p. This interpretation furnishes a simple method of proving certain divisibility theorems concerning binomial coefficients. As examples, explicit expressions for the greatest common divisor and for the least common multiple of $c_{n,1}, c_{n,2}, \cdots, c_{n,n-1}$ are derived.

3. *The other half of euclidean geometry*, by Dr. Kenneth Leisenring, University of Michigan, introduced by the Secretary.

With respect to the undefined terms "point" and "line," euclidean plane geometry is not dual; parallelism is defined for lines but not for points. Since, however, the euclidean plane can be described in purely projective terms, it follows from the duality of the projective plane that there must exist a geometry dual to the euclidean plane, obtainable by dualizing the projective description. In this dual we have a point at infinity rather than a line, and parallelism is defined for points. If we choose as a model for the projective plane the conventional euclidean plane augmented by the line at infinity, we can choose an arbitrary point as the point at infinity in the dual metric, and the two metrics can be simply related to each other. This dual metric can be intuitively grasped and systematically exploited. Results indicated are the obtaining of new theorems by dualization, (which can be repeated at will) and the unification and simplification of metric theorems on conics.

4. *An approximation to the quotient of Gamma functions*, by Professor J. S. Frame, Michigan State College.

A close approximation to the ratio of two nearby values of the Gamma function is obtained by neglecting the remainder $R_n(w)$ in the formula

$$\left(n+\frac{1+w}{2}\right)+\left(n+\frac{1-w}{2}\right)=\left(n^2+\frac{1-w^2}{12}\right)^{w/2}e^{-R_n(w)}.$$

This remainder vanishes for all n when $w=0, \pm 1$, or ± 2. It is less than $1/2(4n)^{-4}$ for $w<1$, so that it may be neglected for $n>2.5$ and $w<1$ if only four significant figures are required. Furthermore, the Gamma function arguments are so chosen that $R_n(w)$ is an even function of n and an odd func-

tion of w. Approximations to the values of certain binomial coefficients and Beta functions follow immediately.

5. *What Ohio colleges are doing about mathematics*, by Professor Wayne Dancer, University of Toledo.

This paper presents the results of a recent questionnaire regarding policies and practices in mathematics in the various Ohio colleges. The study deals with college requirements in mathematics, organization of courses, the proper curriculum for a student majoring in mathematics, poorly prepared students, supervision of instruction, and other problems encountered by the departments of mathematics.

6. *Report of committee on high school mathematics*, by Professor C. C. Richtmeyer, Central Michigan College.

7. *The evaluation of determinants with pivotal methods*, by Professor P. S. Dwyer, University of Michigan.

After a brief discussion of the more general elimination, synthetic, and condensation methods, the treatment was directed to pivotal methods. Special emphasis was given to the exact pivotal method of multiplication and subtraction with exact division. Applications were made to the compact solution of simultaneous linear equations, to determinants whose elements are complex numbers, to determinants whose elements are approximate numbers, to determining all the principal minors of a determinant, and so on. A plea was made for a greater use of pivotal methods in elementary and advanced instruction dealing with the evaluation of determinants.

L. J. ROUSE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-Second Annual Meeting, Columbus, Ohio, December 31, 1948.

Thirty-First Summer Meeting, Boulder, Colorado, September, 1949.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, November 6, 1948.
ILLINOIS, Peoria, May 13–14, 1949
INDIANA
IOWA, Drake University, Des Moines, April 15–16, 1949
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, Spring, 1949
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May, 1949
NORTHERN CALIFORNIA, San Francisco, January 29, 1949
OHIO, Ohio State University, Columbus, April 2, 1949
OKLAHOMA
PACIFIC NORTHWEST, Oregon State College, Corvallis, Spring, 1949
PHILADELPHIA, Philadelphia, November 27, 1948
ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949
SOUTHEASTERN, University of Alabama, University, March 18–19, 1949
SOUTHERN CALIFORNIA, John Muir Jr. College, Pasadena, March 12, 1949
SOUTHWESTERN
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UPPER NEW YORK STATE, University of Buffalo, May, 1949
WISCONSIN, Lawrence College, Appleton, May 14, 1949

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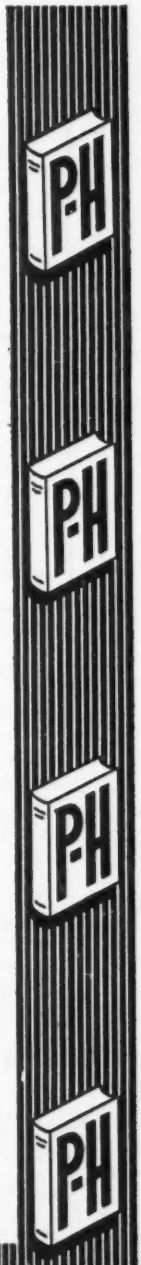

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